

INTENTIONAL NONLINEARITY IN A
STATE VARIABLE FEEDBACK SYSTEM

by

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Date

DEDICATION

This thesis is dedicated to my parents
and all those who have inspired me in my achievements.

ACKNOWLEDGMENT

The author wishes to acknowledge the encouragement, advice, and suggestions rendered by Professor Donald D. Schultz and Mr. Charles R. Slivinsky in the preparation of this thesis.

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ABSTRACT

In this thesis a particular type of nonlinear state variable feedback system is discussed. The system contains a single nonlinearity, and it is shown by describing function techniques and examples that the optimum location of the nonlinear element for maximum control is at the left end. A method for designing gain-insensitive systems is presented, and it is shown by simple reasoning and examples that the system response for the gain-insensitive design is better than that of systems designed by conventional state variable feedback.

A method is given to overcome the effects of saturation within the fixed plant by introducing an intentional nonlinearity to limit the saturating elements to their linear regions of operation. This makes it possible to apply the above gain-insensitive design technique so that the nonlinear plant can be made absolutely stable for all gain. The proposed method is then applied to improve the response of a fuel valve servomechanism, and the system is evaluated using an analog computer.

CHAPTER I

INTRODUCTION

Throughout the course of scientific and industrial development the problem of controlling dynamical processes has existed. However, control engineering, considered as one of the engineering disciplines, is less than a century old. After World War II complex industrial processes and especially the modern space program have boosted the demand for automation and control engineering science.

The problem of the control engineer is to control a plant so that in spite of the expected variation in system description, the behavior of the system does not exceed predetermined limits. Stringent performance requirements and the sophisticated devices of modern technology have resulted in the development of complicated systems which cannot be synthesized and analyzed by linear techniques. In addition to inherent system nonlinearities sometimes nonlinearities are introduced intentionally to realize better performance or more economy. The investigation and development

of methods of introducing an intentional nonlinearity to improve the system performance is the subject of this work.

History

Many systems can be described by differential equations which may be linear or nonlinear. When the differential equations are linear, the system is also said to be linear and can be designed by the frequency domain as well as the time-domain approach. In the frequency-domain Laplace transform techniques are used for analysis and synthesis; improvement in the system performance is usually effected by series and feedback compensation designed with the aid of Bode diagrams (Bower and Shultheiss, 1959). An alternate method of design uses root locus techniques (Truxal, 1956). The problem of determining stability can be solved by such well-known methods as Nyquist diagrams, root locus techniques, and the Routh-Hurwitz criteria. For nonlinear systems the differential equations describing the system are nonlinear, and Laplace transform techniques are not applicable; no direct, analytical approach exists to relate the input and the output of the system. Some of the more widely known methods for the analysis and synthesis of nonlinear

systems are linearization about an operating point, graphical techniques such as the isocline method (Thaler and Pastel, 1963), describing function methods (Gibson, 1963), and analytical and numerical solutions. For stability investigations one can use the second method of Liapunov (Schultz, 1965), the describing function method, and Popov criteria.

The modern approach to control system design is quite different from conventional approaches. A system must be described by n first-order differential equations defining n state variables. The design is accomplished by feeding back all state variables after multiplication by constant coefficients. Here the system is controlled by all the states and Schultz and Melsa (1967) have shown that a linear system can be optimized for a quadratic Performance Index by feeding back a linear combination of all variables. In addition, any desired closed-loop response can be achieved by feeding back all the state variables in the proper combination. Herring (1967) has shown that a certain class of nonlinear systems can be made absolutely stable and gain-insensitive by feeding back all the state variables.

Outline of Thesis

In the following chapters it is shown how the introduction of an intentional nonlinearity can be combined with state variable feedback to overcome the effects of saturation. A step-by-step development is presented with illustrative examples, and the method is applied to improve the response of a practical problem.

Chapter II deals with the representation of linear and nonlinear state variable feedback systems. Stability criteria for nonlinear systems are presented along with a brief description of describing function theory. The effect of the location of the nonlinearity is investigated, and it is concluded that the optimum location is at the left most end for maximum control over the system. Finally, the chapter is concluded with illustrative examples.

In Chapter III the effect of saturation in a system is discussed and the idea of introducing an intentional, saturation type of nonlinearity is described. The concept of gain-insensitive systems is presented for linear as well as nonlinear systems. Two systems, gain-insensitive and non-gain-insensitive, are compared and discussed. It is shown that the gain-insensitive system

is absolutely stable and has a satisfactory step response when the gain is varied or operates in the nonlinear region. Finally, a design technique is given for overcoming the effects of saturation by introducing an intentional nonlinearity.

In Chapter IV the techniques developed in Chapter II and III are applied to improve the response of a fuel valve servomechanism. The design is evaluated using both digital and analog computers, and the results are presented in recorded form.

The final chapter presents the conclusions and suggestions for further investigation.

CHAPTER II

GENERAL THEORY

In this chapter the modern representation of linear systems is discussed and state variable feedback methods are presented; general expressions for the transfer functions $G_{eq}(s)$, $H_{eq}(s)$, $Y/R(s)$, etc., are given in matrix form. It is shown that for systems which contain a single nonlinearity but are otherwise linear, the corresponding expressions for $G_{eq}(s)$, $H_{eq}(s)$, etc., depend on the location of the nonlinearity in the forward path.

The effects of the location of the nonlinear element in a system are further investigated by applying describing function theory; and it is concluded that, when the nonlinearity is located at the left end of the system, desirable stability properties and maximum control over the system are achieved. Finally, the results are illustrated with a third-order system having a single nonlinearity.

Representation of Linear Systems

There are two different ways to represent control systems: the input-output form and the modern

state variable method. Here the latter one is of prime interest and hence is presented in detail.

Consider Fig. 2-1, showing a general representation of a linear system. G_p represents the plant and is described by the following set of n first-order linear differential equations:

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{b}u \quad (\text{Ab})$$

$$y = \underline{c}^T \underline{x} \quad (\text{c})$$

where

\underline{A} is the $n \times n$ plant matrix

\underline{b} is the $n \times 1$ control vector

\underline{c} is the $n \times 1$ output vector

\underline{k} is the $n \times 1$ column matrix of feedback coefficients

\underline{x} is the $n \times 1$ state vector

u is the input to the plant

r is the input to the system

\underline{I} is the $n \times n$ identity matrix

The transfer function $G_p(s)$ relating the control function u and the output of the system y is given by

$$G_p(s) = y/u(s) = \underline{c}^T \underline{\Phi}(s) \underline{b} \quad (2-1)$$

where $\underline{\Phi}(s)$ is called the resolvent matrix and is given by $(s\underline{I} - \underline{A})^{-1}$ (Schultz and Melsa, 1967). The input and output of the system are related by

$$Y/R(s) = \underline{c}^T \underline{\Phi}_K \underline{b} \quad (2-2)$$

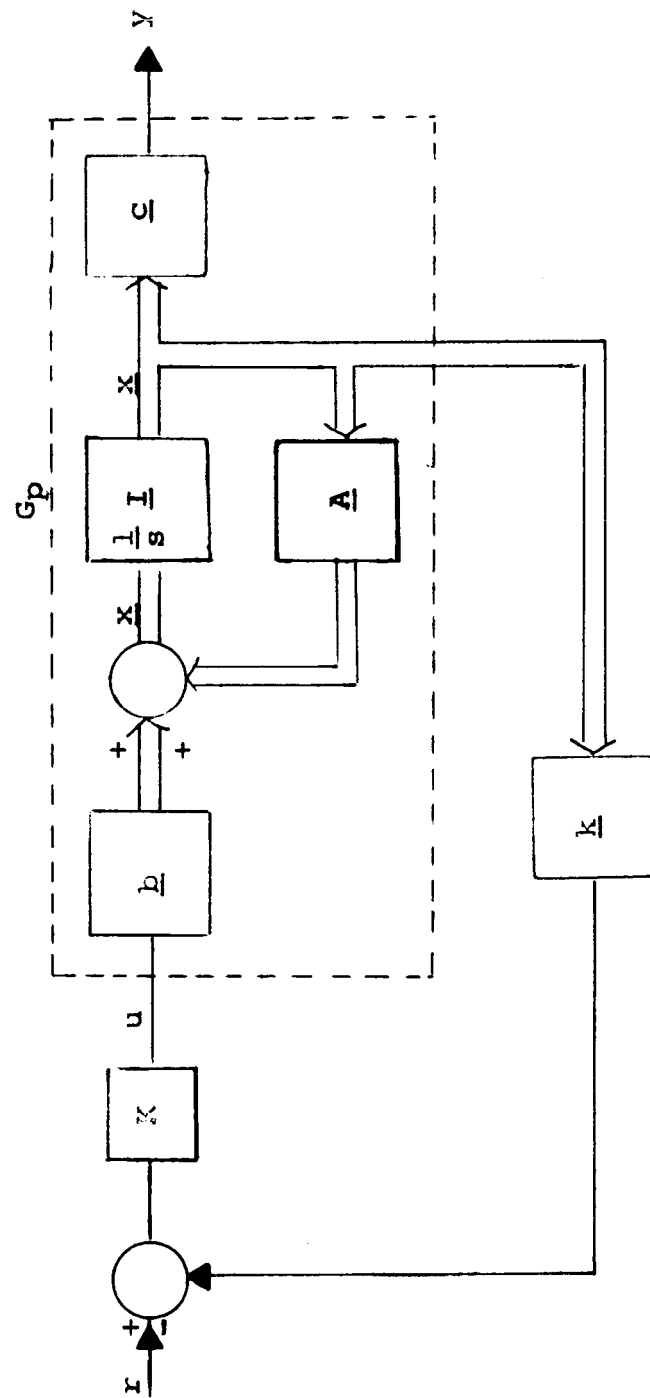


Fig. 2-1 Basic Configuration for a Linear State Variable Feedback System ∞

where Φ_k is the closed-loop resolvent matrix, given by $(sI - A_k)^{-1}$ or $(sI - A + bk^T)^{-1}$.

The system of Fig. 2-1 can be represented by two alternate block diagram configurations, the $G_{eq}(s)$ and $H_{eq}(s)$ representations shown in Fig. 2-2(a) and (b), respectively. General expressions for $H_{eq}(s)$, $G_{eq}(s)$, and $G_p(s)H_{eq}(s)$ are given below.

$$H_{eq}(s) = (k^T \Phi(s) b) / c^T \Phi(s) b \quad (2-3)$$

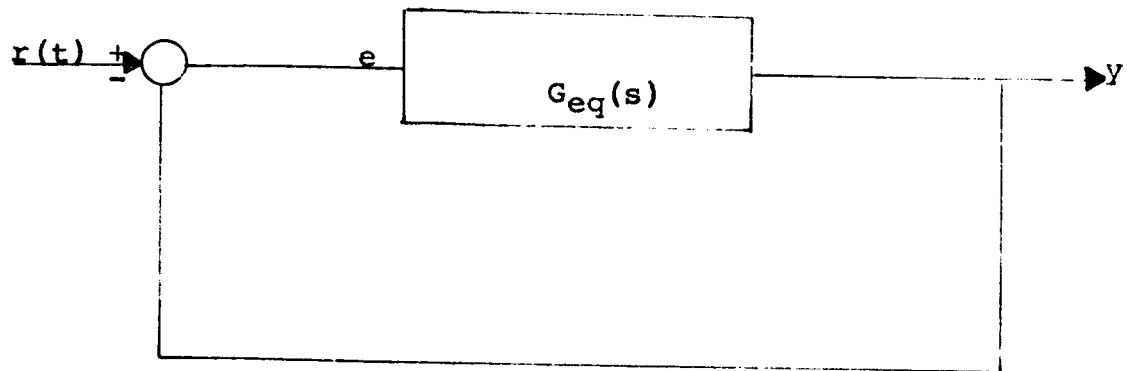
$$G_{eq}(s) = (c^T \Phi(s) b) / (1 + (k - c)^T \Phi(s) b) \quad (2-4)$$

$$G_p H_{eq}(s) = k^T \Phi(s) b \quad (2-5)$$

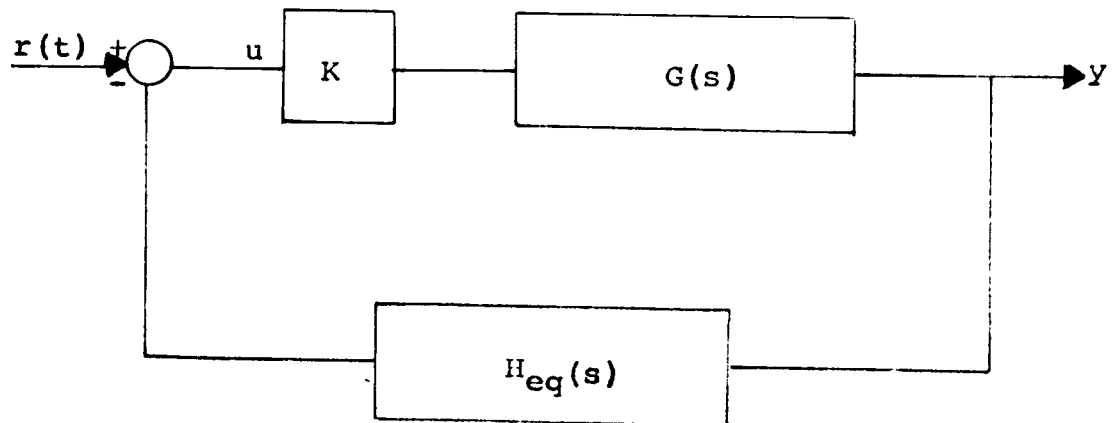
All the above expressions can be found in terms of Φ_k (Schultz and Melsa, 1967).

State Variable Representation of a Particular Type of Nonlinear System

Consider the configuration shown in Fig. 2-3 and having the single nonlinearity represented by the block labelled N. G^1, G^2, \dots, G^n each represents a first-order transfer function; i.e., $G^i = k_i(s + z_i)/(s + p_i)$. Block diagram manipulation yields the modified diagram shown in Fig. 2-4(a), where $G_1(s)$ and $G_2(s)$ represent $(n - i)^{th}$ and i^{th} -order linear transfer functions, respectively, and $H_{1eq}(s)$ and $H_{2eq}(s)$ are of order $(n - i - 1)$ and $(i - 1)$, respectively. Further



(a)



(b)

Fig. 2-2 G_{eq} and H_{eq} Method of Representing a Linear System

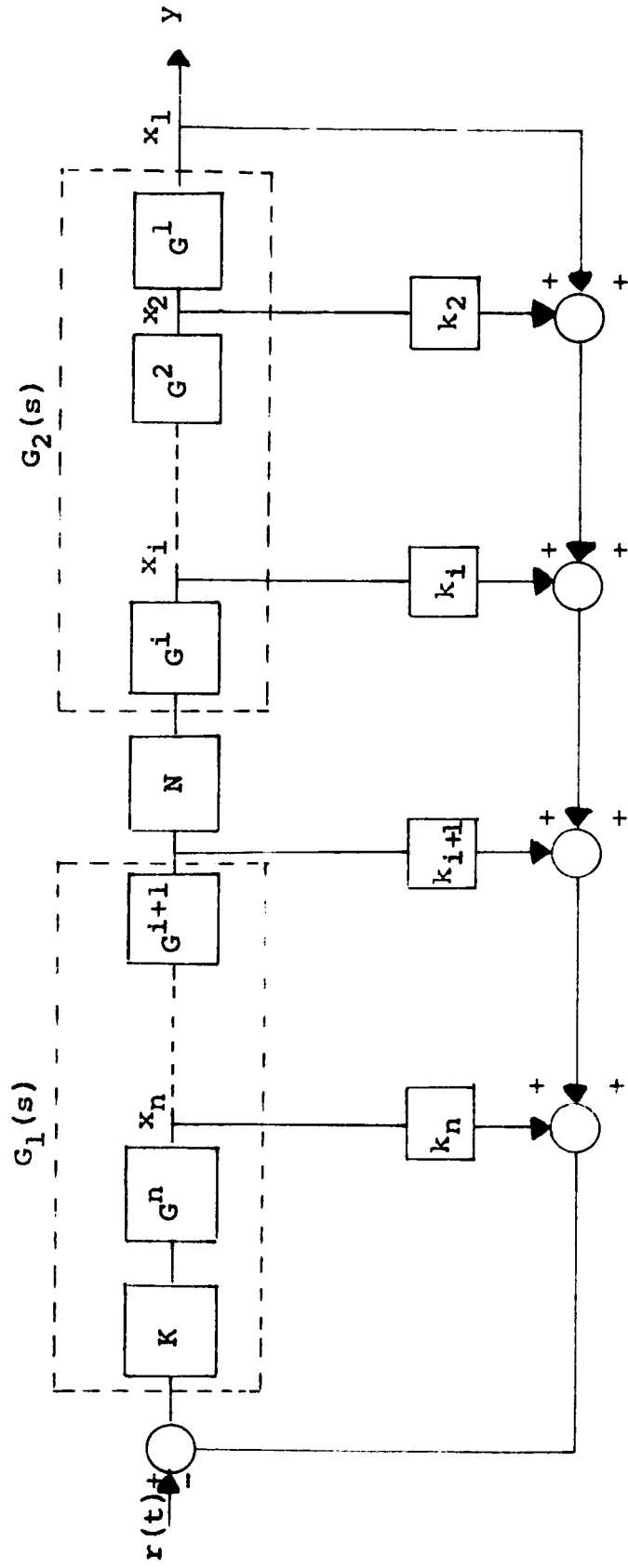
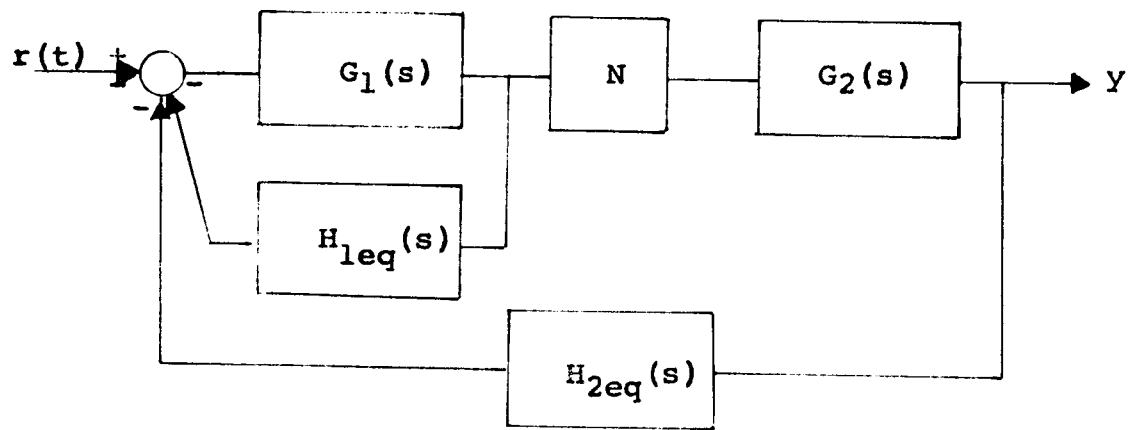
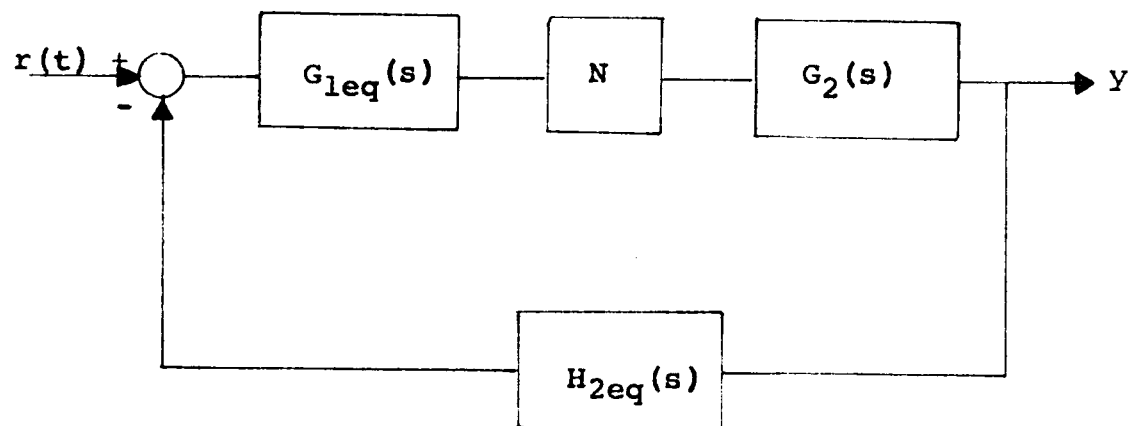


Fig. 2-3 A State Variable Feedback System with the Nonlinearity N in the Forward Path



(a)



(b)

Fig. 2-4 Block Diagram Reduction for the System Shown in Fig. 2-3

reduction of the block diagram shows the system in final form in Fig. 2-4(b).

Now comparing the representations for linear systems and nonlinear systems, one can observe that nonlinear systems cannot be represented in the simplest G_{eq} or H_{eq} form (as can linear systems) unless the nonlinearity is located at the left most end. In the general case (see Fig. 2-4(b)) linear transfer functions and characteristics of the nonlinearity are required to describe the nonlinear system. $H_{eq}(s)$ has $n - 1$ zeros while $H_{2eq}(s)$ has $1 - 1$ zeros. As the nonlinearity is shifted towards the left side, the number of zeros in $H_{2eq}(s)$ increases and finally becomes $n - 1$ when it is located at the left end.

Describing Function Theory

The describing function method is based on an analysis which neglects the effects of harmonics in the system, so that the accuracy of technique increases with the order of the system. The system configuration shown in Fig. 2-5 represents the reduced form of Fig. 2-4(b) and is in the correct form for applying the describing function method. N is the single nonlinearity of the system and is assumed to be insensitive

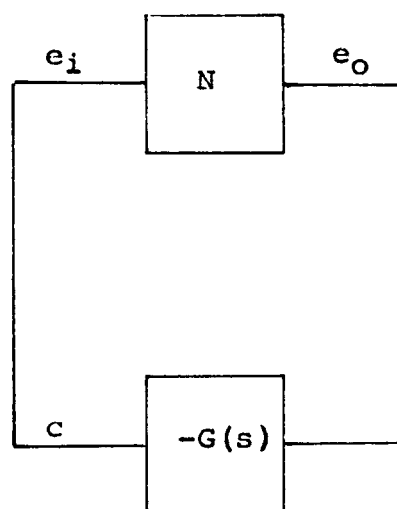


Fig. 2-5 Equivalent or Reduced Form
of Fig. 2-4(b) with $r(t) = 0$

to frequency. It is desired to determine whether a sustained oscillation of sinusoidal form exists in the system when there is no external input.

The output of the nonlinear element when its input is a sinusoidal wave having an amplitude E is written in the form

$$e_o = k_{eq}e + f_d(e) \quad (2-6a)$$

The first term on the right-hand side is the fundamental while the second term represents harmonic distortion and is neglected. Hence

$$e_o \approx k_{eq}e \quad (2-6b)$$

k_{eq} is known as the equivalent gain, or the describing function, and it is a function of input-signal amplitude E . The describing function for the non-linearity can be found as follows (Gibson, 1963):

$$k_{eq} = g(E) + jb(E) \quad (2-7)$$

where

$$g(E) = \frac{1}{\pi E} \int_0^{2\pi} f(E \sin \theta) \sin \theta d\theta \quad (2-8a)$$

$$b(E) = \frac{1}{\pi E} \int_0^{2\pi} f(E \sin \theta) \cos \theta d\theta \quad (2-8b)$$

From Fig. 2-5

$$\frac{c(j\omega)}{e_o(j\omega)} = -G(j\omega) \quad (2-9)$$

Referring to the equations (2-6b) and (2-9), one can see that for the existence of sustained oscillations

there must exist a simultaneous solution which satisfies both equations; i.e.,

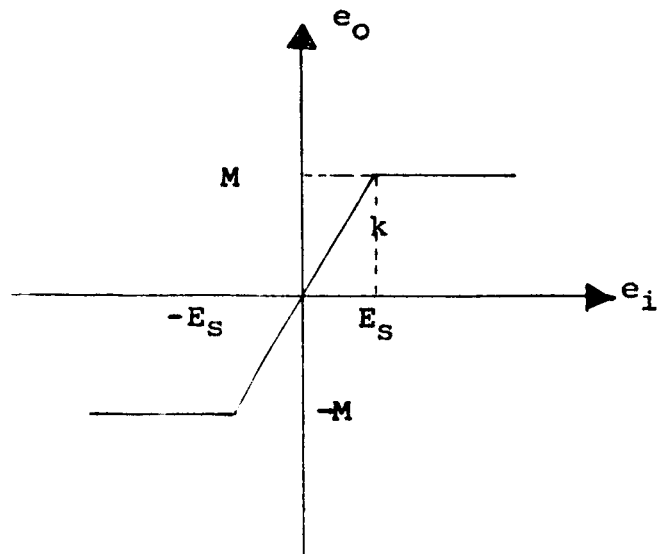
$$G(j\omega) = -\frac{1}{k_{eq}} \quad (2-10)$$

A convenient way of investigating equation (2-10) is to draw polar plots of both sides and check for an intersection; the point of intersection gives the frequency and amplitude of oscillation. The oscillations may be stable or unstable depending on whether the amplitude of oscillation decreases or increases as the operating point on $-1/k_{eq}$ locus moves within the frequency-sensitive locus of $G(j\omega)$; i.e., the Nyquist plot.

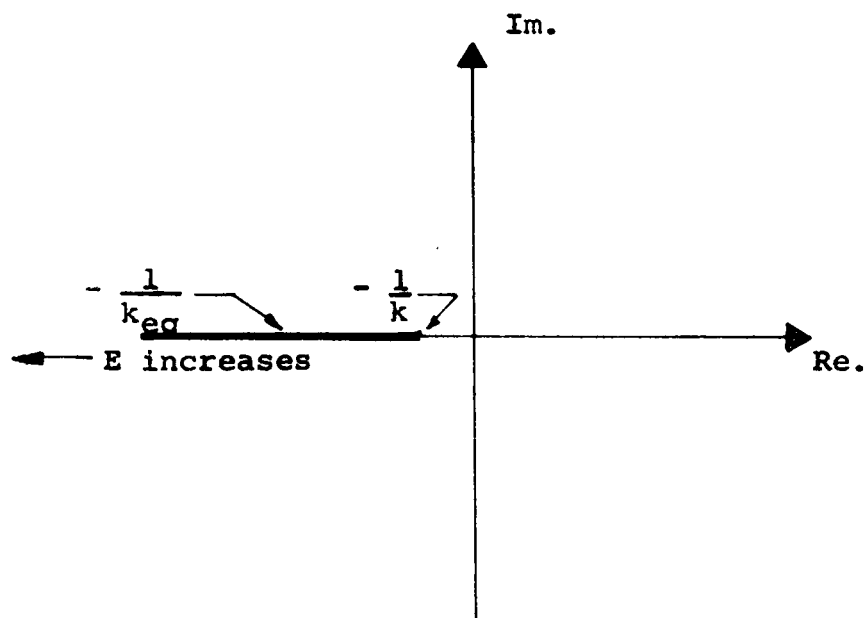
One can apply the describing function method to check the stability of a system having a particular type of nonlinearity N . N is single-valued and symmetric, lying in first and third quadrants. The describing function for this type nonlinearity will always be real and non-negative (Gibson, 1963). Fig. 2-6(a) shows a saturation type nonlinearity, a representative of the class we are considering. The equivalent gain for such a nonlinearity is given by (Thaler and Pastel, 1962)

$$k_{eq} = \frac{2k}{\pi} \left(\sin^{-1} \frac{E_s}{E} + \frac{E_s}{E} \sqrt{\left(1 - \frac{E_s}{E}\right)^2} \right) \angle 0 \quad (2-11)$$

which is always real and non-negative, as expected. The polar plot is shown in Fig. 2-6(b).



(a)



(b)

Fig. 2-6 Characteristic and the Polar Plot of Equivalent Gain for the Nonlinearity

Consider the system whose block diagram is shown in Fig. 2-7(a) which is similar to Fig. 2-4(b). It was stated previously that $G_{1eq}(s)$ has $n - 1$ poles, $G_2(s)$ has i poles, and $H_{2eq}(s)$ has $i - 1$ zeros. Hence

$$G(j\omega) = G_{1eq}(s)G_2(s)H_{2eq}(s) \quad (2-12)$$

has n poles and $i - 1$ zeros. Now to check for the existence of oscillations, the polar plot of $-\frac{1}{k_{eq}}$ for a single-valued, symmetric nonlinearity is plotted in Fig. 2-8. For oscillations

$$G(j\omega) \Big|_{\omega = \omega_c} \leq -\frac{1}{k} \quad (2-13)$$

where ω_c is a frequency for which $G(j\omega)$ is real. This is possible if and only if $G(j\omega)$ is inherently unstable in the linear region or $G(j\omega)$ is conditionally stable as shown in Fig. 2-8, labelled $G''(j\omega)$ and $G'(j\omega)$, respectively.

From Equation (2-13) it can be seen that oscillations can exist for some value of gain k as long as the polar plot of $G(j\omega)$ crosses the negative real axis. Thus to avoid oscillations $G(j\omega)$ should not cross the negative real axis for any value of gain; i.e., $G(j\omega)$ should have the form shown by the curve in Fig. 2-8 and labelled $G'''(j\omega)$. This is possible if $G(j\omega)$ has a pole-zero excess of ≤ 2 and if the zeros of $G(j\omega)$ are located at proper places. Thus it is desired to have

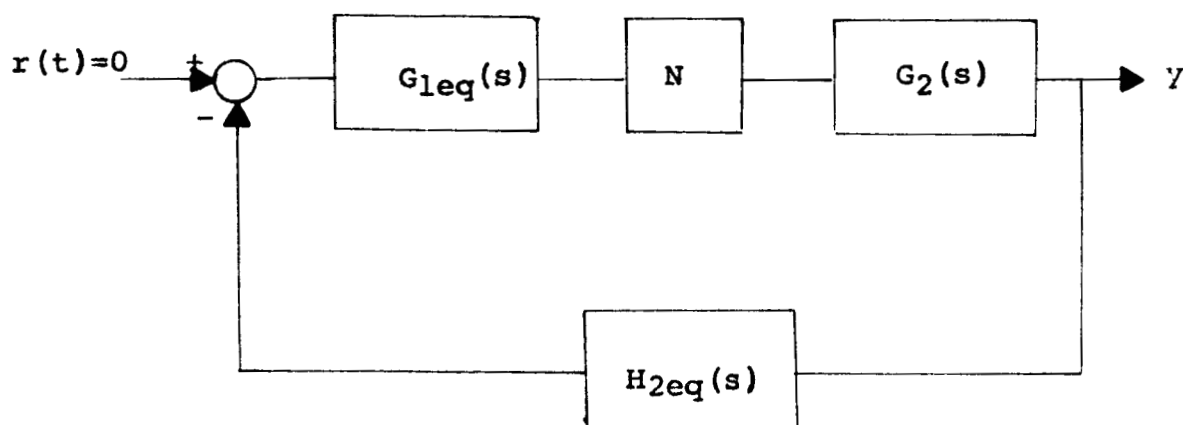


Fig. 2-7 The System of Fig. 2-4(b)

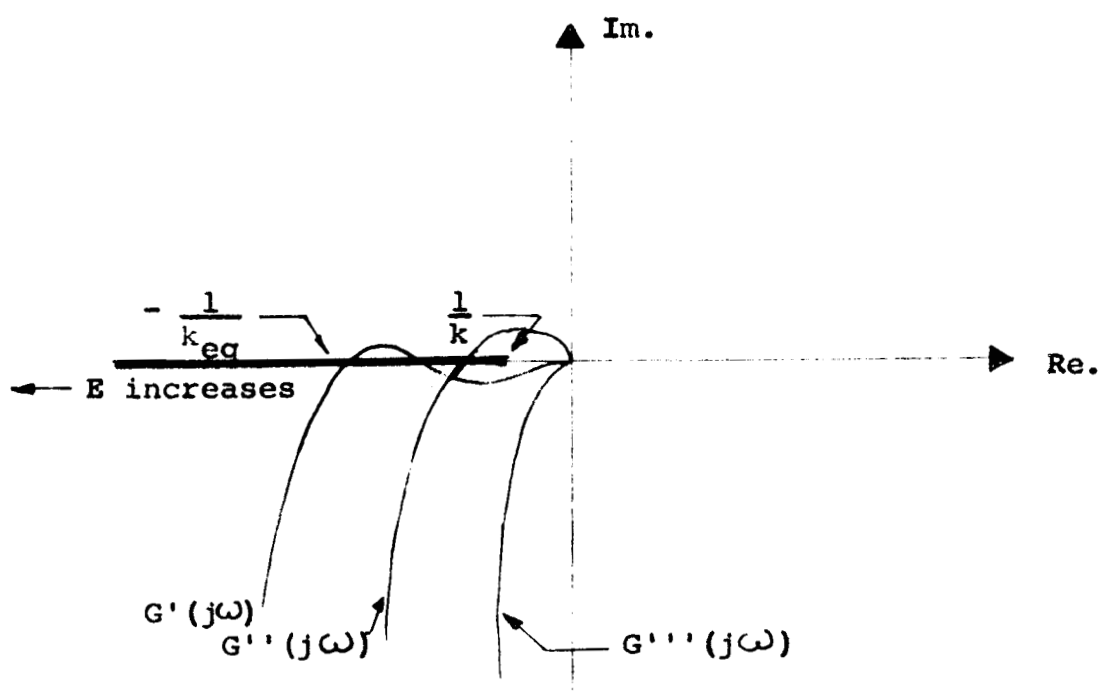


Fig. 2-8 Various Types of $G(j\omega)$ Functions Showing the Possibility of Oscillations and the Polar Plot of $-1/k_{eq}$ for the Nonlinearity

$$\begin{aligned} n - i + 1 &\leq 2 \\ i &\geq n - 1 \end{aligned} \quad (2-14)$$

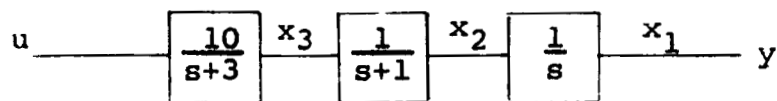
in order to prevent oscillations. Also, it is known that the more zeros there are in $H_{2eq}(s)$, the better a system can be controlled, so that the optimum choice for i would be n ; that is, the best location for the nonlinearity is at the left most end of the system. It should be noted that stability of the system still depends upon the zeros of $H_{2eq}(s)$ and hence the feedback coefficients.

Example

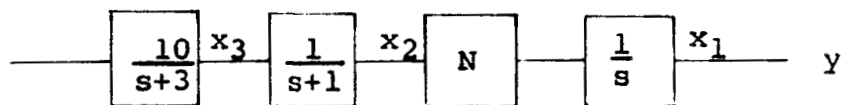
Consider the plant shown in Fig. 2-9(a) which is to be controlled by state variable feedback. All systems saturate at one or another point. Here saturation is accounted for by the nonlinearity labelled N , which is presumed to be of the type shown in Fig. 2-6(a). Different possibilities for saturation are shown in Fig. 2-9(b), (c), and (d). It is the purpose of this example to investigate what happens when the system saturates at these different points.

Case I

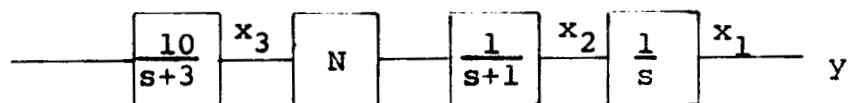
Let N be located as in Fig. 2-9(b). State variable feedback is to be used to achieve the closed-loop transfer function



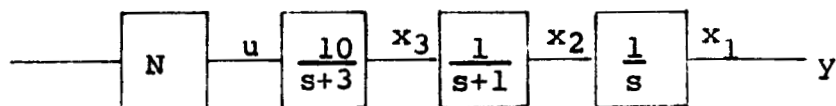
(a)



(b)



(c)



(d)

Fig. 2-9 Plant Showing Saturation at Different Points in the System

$$\frac{Y}{R}(s) = \frac{10}{s^3 + 5.25s^2 + 8s + 10}$$

when operating in the linear region. The result is shown in Fig. 2-10(a). When the system operates in the nonlinear region, the input-output relation does not hold; but some aspects of the behavior can be investigated by the describing function method. By block diagram reduction of Fig. 2-10(a)

$$G_{1eq} = \frac{10}{s^2 + 5.25s + 8}$$

$$G_2(s) = \frac{1}{s}$$

$$H_{2eq}(s) = 1$$

so that

$$\begin{aligned} G(s) &= G_{1eq}(s) \cdot H_{2eq}(s) \cdot G_2(s) \\ &= \frac{10}{s(s^2 + 5.25s + 8)} \end{aligned}$$

The polar plots of $G(j\omega)$ and $-\frac{1}{k_{eq}}$ as given by Equation (2-12) are shown in Fig. 2-11. The point at which $G(j\omega)$ intersects with the negative real axis can be found very easily to be -0.238 at $\omega = 2\sqrt{2}$.

That is,

$$\begin{aligned} G(j\omega) \bigg|_{\omega = 2\sqrt{2}} &= \frac{10}{2\sqrt{2}(-8 + j5.25 \cdot 2\sqrt{2} + 8)} \\ &= -0.238 \end{aligned}$$

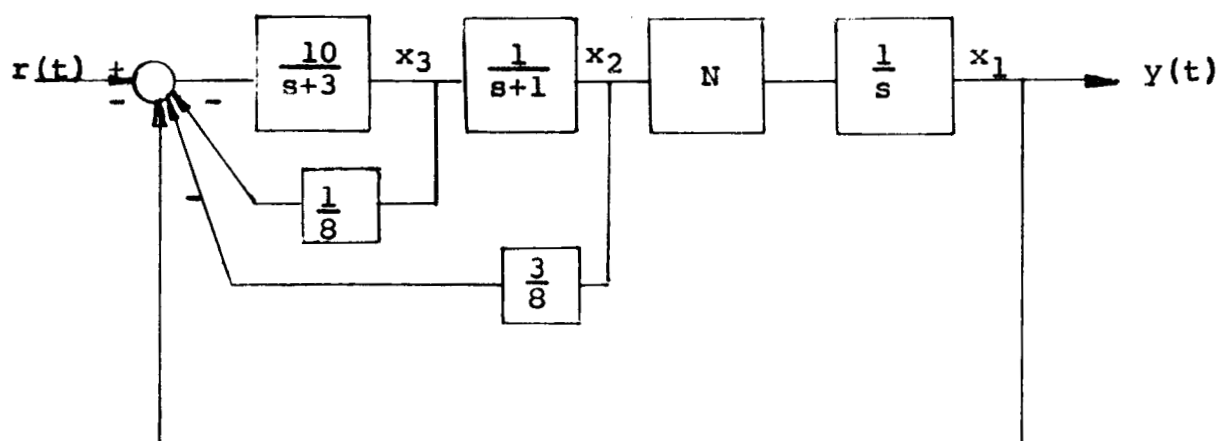


Fig. 2-10 Nonlinear System Designed by
State Variable Feedback Method

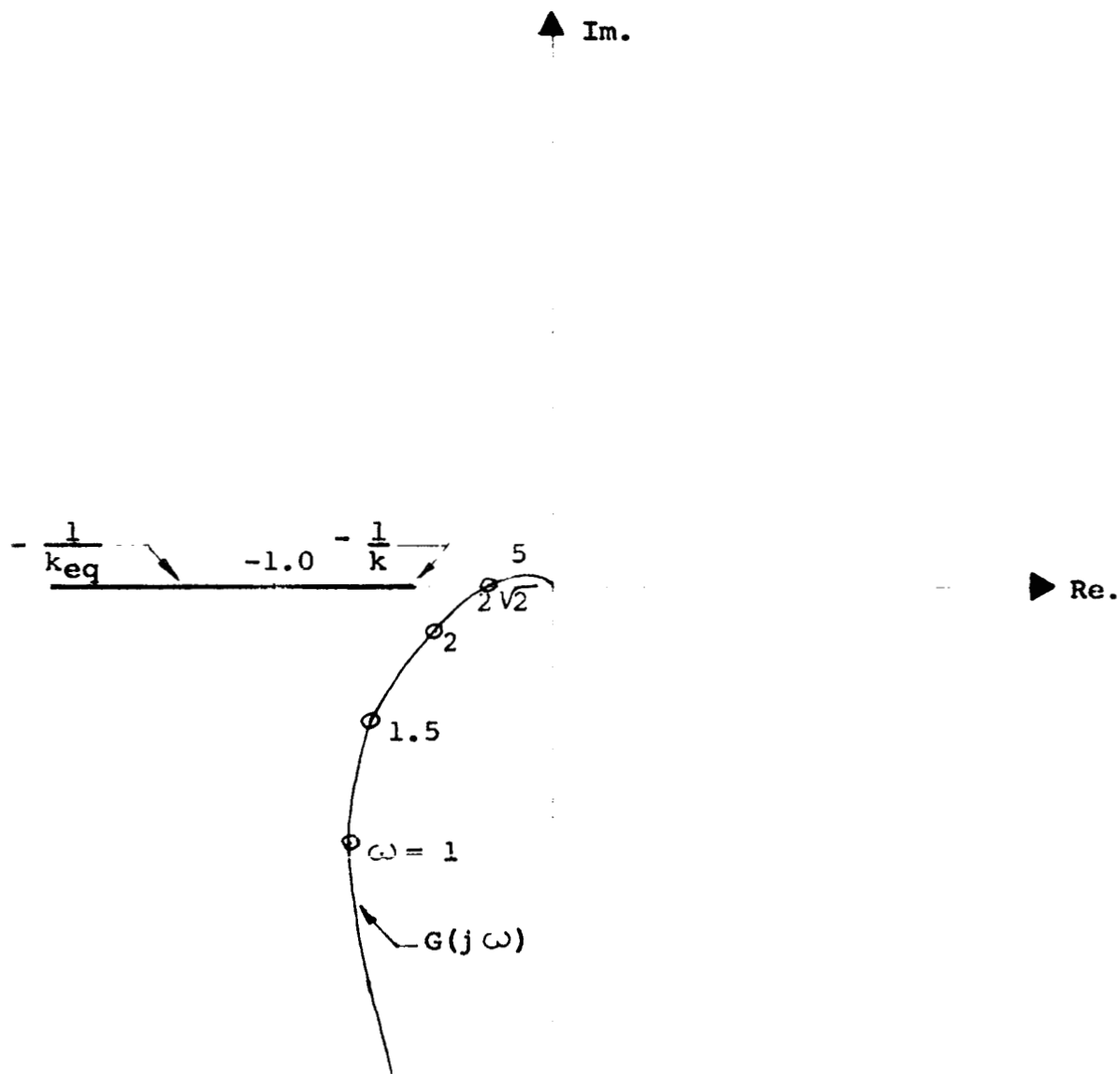


Fig. 2-11 Polar Plot of $G(j\omega)$ and $-1/k_{eq}$ for Case I

Thus for oscillation $(-\frac{1}{k_{eq}})_{\max} = -0.238$, which gives the maximum value of k , the linear gain, which the nonlinearity can have. In this case oscillations will occur when k is increased beyond $1/0.238$; however, examples can be found where even without variation of k , the system can show oscillations. One such system is shown in Fig. 2-12(a) along with its polar plot in Fig. 2-12(b).

Case II

Let N be located as in Fig. 2-9(c). The system still has the same configuration when operating in the linear region. When operating in the nonlinear region

$$G_{1eq}(s) = \frac{10}{s + 4.25}$$

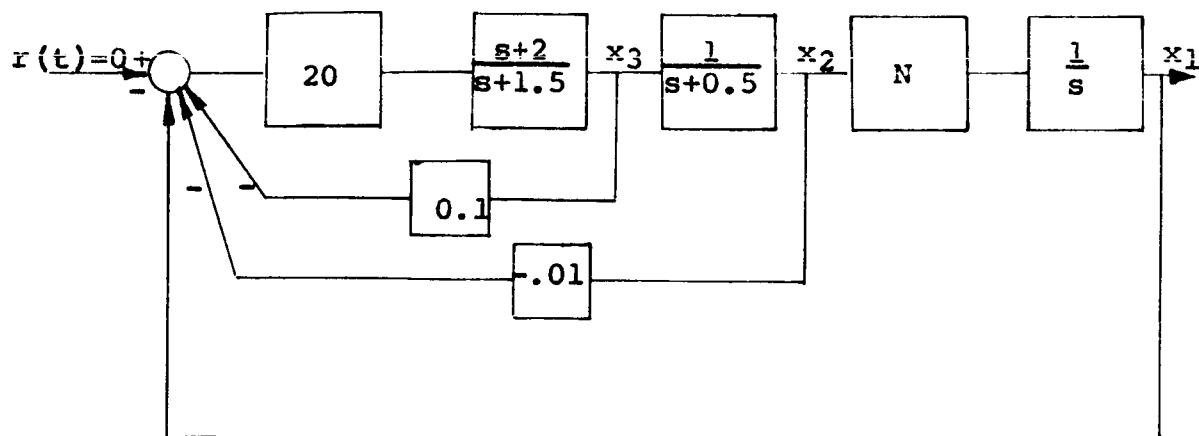
$$H_{2eq}(s) = \frac{3}{8}(s + 2.66)$$

$$G_2(s) = \frac{1}{s(s + 1)}$$

so that

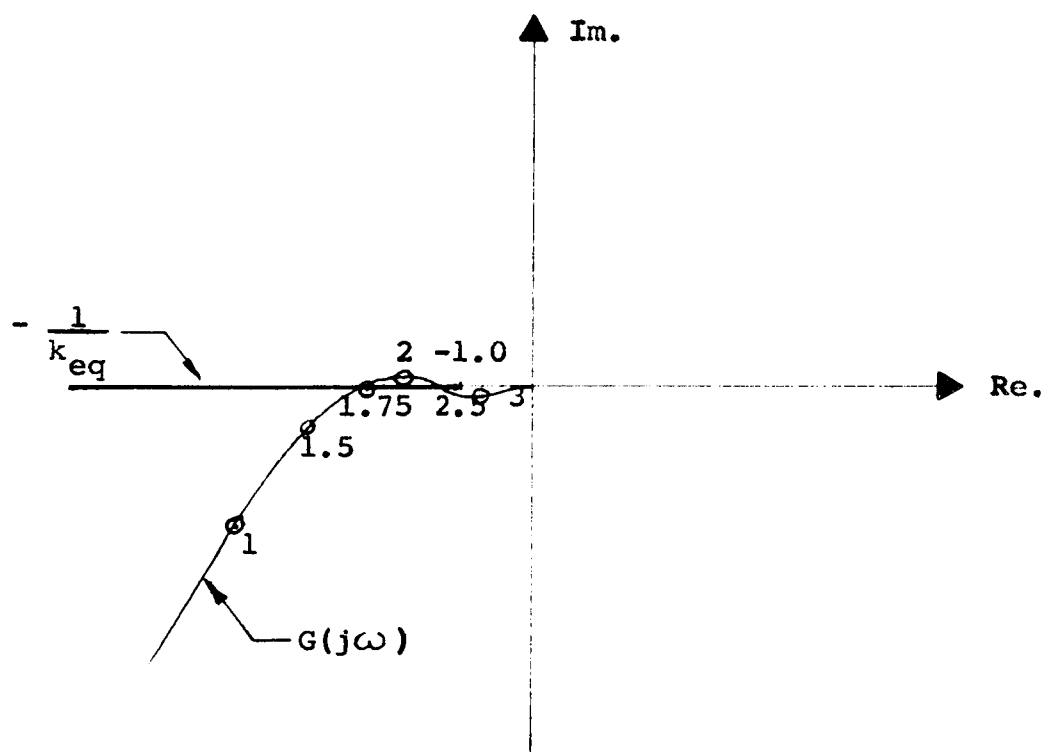
$$G(s) = \frac{3.75(s + 2.66)}{s(s + 1)(s + 4.25)}$$

The polar plot for $G(j\omega)$ and $-\frac{1}{k_{eq}}$ are shown in Fig. 2-13, and it can be seen that there cannot be an intersection for any value of gain k of the nonlinearity or for any gain associated with $G(j\omega)$. Thus there is no oscillation and the system is stable for all gain.



$$G(s) = \frac{20(s+2)}{3s(s^2 + 2.0266s + 0.783)}$$

(a)



(b)

Fig. 2-12 Polar Plot of $G(j\omega)$ and $-1/k_{eq}$ for the System Shown in Fig. 2-12(a)

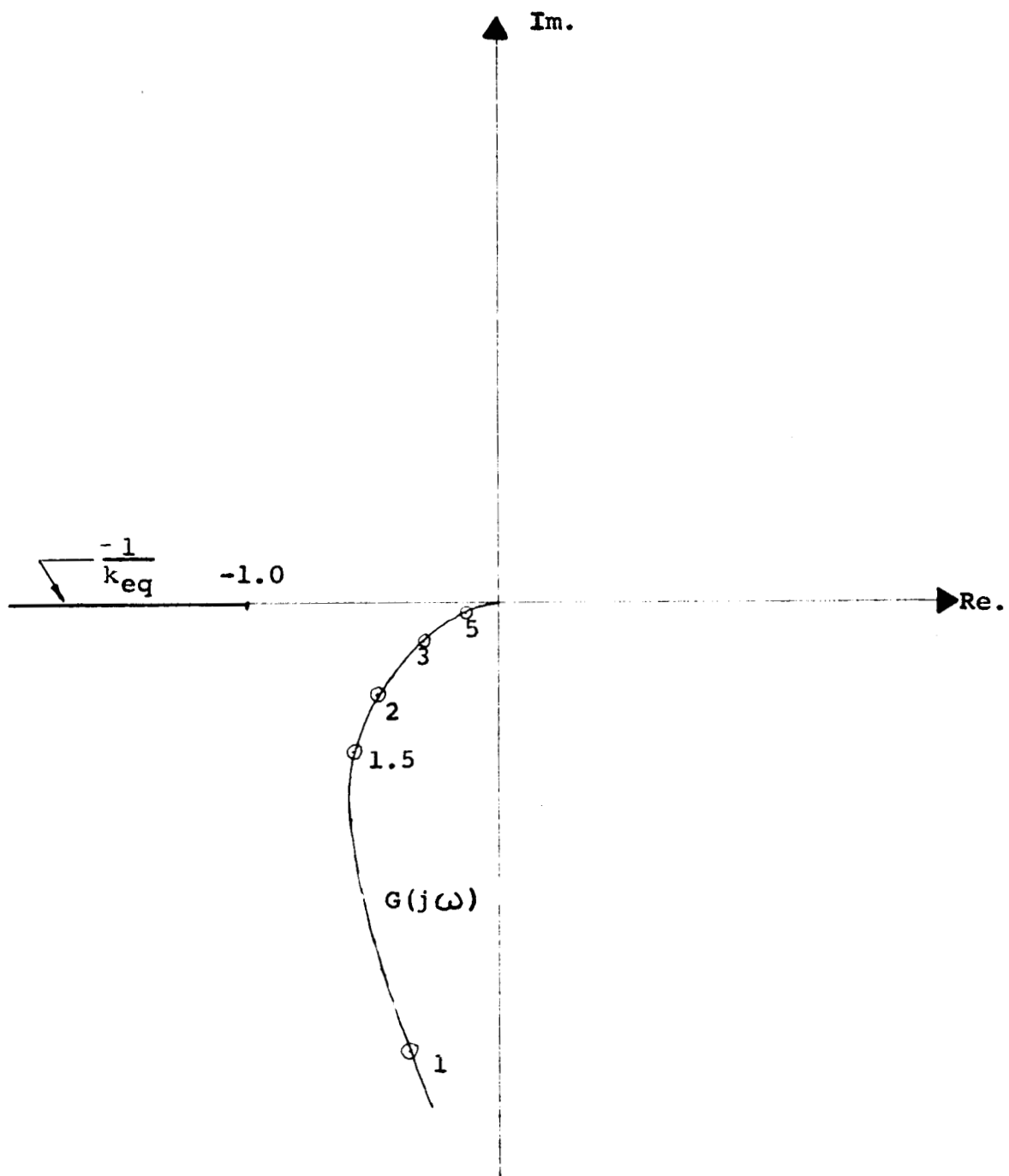


Fig. 2-13 Polar Plot of $G(j\omega)$ and $-1/k_{eq}$ for Case II

Case III

Let N be located as in Fig. 2-9(d). In the nonlinear region

$$G_{1eq}(s) = 1$$

$$G_2(s) = \frac{10}{s(s+1)(s+3)}$$

$$H_{2eq}(s) = \frac{1}{8}(s^2 + 4s + 8)$$

so that

$$G(s) = \frac{1.25(s^2 + 4s + 8)}{s(s+3)(s+1)}$$

Again, it can be seen from the polar plot of Fig. 2-14 that the system is stable for all gain whether it be associated with the nonlinearity or with any other gain in the forward loop.

Comparing all three cases, one can see that as the N is moved towards the left end the number of zeros of $H_{2eq}(s)$ increases, forcing the polar plot of $G(j\omega)$ to approach the origin at a lower multiple of 90° . Finally, when saturation takes place at the left most state variable, $G(j\omega)$ approaches the origin at -90° , and the example system becomes stable for all gain. Still, placing nonlinearity at the left end does not give assurance of stability if the system is conditionally stable in the linear region, as the location of zeros of

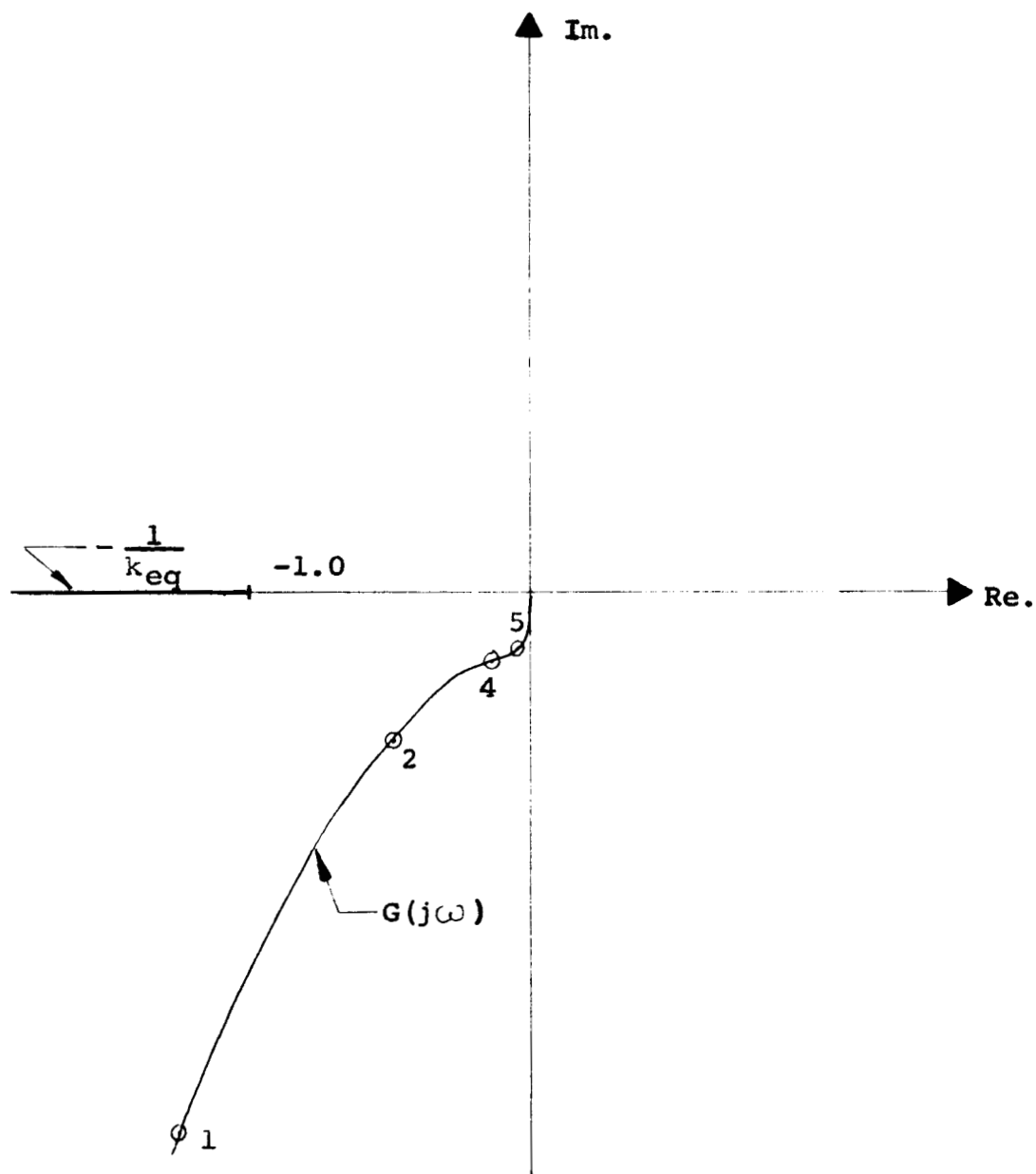


Fig. 2-14 Polar Plot of $G(j\omega)$ and $-1/k_{eq}$ for Case III

$H_{2eq}(s)$ influences the shape of the polar plot of $G(j\omega)$ and hence helps to determine whether or not there are any intersections with the plot of $-\frac{1}{k_{eq}}$.

To assure the absolute stability for all values of gain, a method of designing a system is presented in the next chapter. Thus it can be concluded if N is located at the left end, the number of zeros of $H_{2eq}(s)$ to control the system is at a maximum; and the system can be made stable for all gain by placing these zeros at proper places.

Although the conclusions derived above were discussed for the system having a saturation type nonlinearity, they also hold for any frequency-insensitive, single-valued, and symmetrical nonlinearity, as k_{eq} for such nonlinearity is always real and non-negative.

CHAPTER III

DESIGN OF NONLINEAR GAIN-INSENSITIVE SYSTEMS

In Chapter II it was shown that the stability of systems containing a single nonlinearity and designed by using state variable feedback depends upon the location of both the nonlinearity and the zeros of $H_{2eq}(s)$. In this chapter the same type of system is studied further and a method of making the system gain-insensitive to ensure stability is presented. Systems designed by the proposed method are shown to have absolute stability for any value of gain associated with the linear part of the system or with the nonlinearity.

Next, gain-insensitive and non-gain-insensitive systems having the same closed-loop transfer function in the linear region are compared and significant features of gain-insensitive systems are presented. One can show how the introduction of an additional intentional nonlinearity and state variable feedback can be combined to design systems to have both absolute stability and satisfactory transient response. The technique utilizes the results of Herring (1967), who

has suggested a method of designing systems which are absolutely stable for all values of gain. He has shown that a system can be made absolutely stable and insensitive to gain if $n - 1$ of the n open-loop poles are placed where $n - 1$ of the n closed-loop poles are required. In other words, in terms of Fig. 3-1, the zeros of $H_{eq}(s)$ are placed at the same places where $n - 1$ of the n poles of $G(s)$ are located.

A step-wise procedure for designing a gain-insensitive system is given below.

1. Describe the system in physical variables and assume all the variables are available for control purposes.
2. Choose the desired locations of the n closed-loop poles of Y/R .
3. Modify the plant, or open-loop system, with series or feedback compensation such that $n - 1$ open-loop poles are located at the positions of $n - 1$ of the desired poles of Y/R .
4. Use state variable feedback to force the $n - 1$ zeros of H_{eq} to coincide with $n - 1$ of the new poles of $G(s)$.

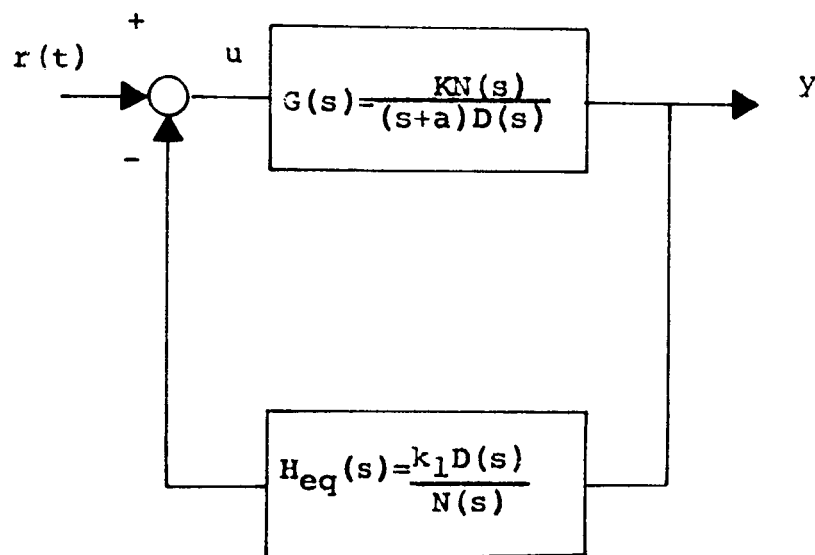


Fig. 3-1 A Linear Gain-Insensitive System,
Where $G(s)H_{eq}(s) = k'/s+a$

5. If all the variables are not available, use the calculated values of the feedback coefficients to determine the required minor-loop compensation (Schultz and Melsa, 1967).

A system designed by the gain-insensitive method has only 1 out of the n closed-loop poles as a function of gain, whereas a non-gain-insensitive system has all n of its closed-loop poles as a function gain. Thus when the gain varies, the response of the gain-insensitive system is likely to change very little; however, the response of the non-gain-insensitive system can change significantly, and the system may even become unstable. Also, the gain-insensitive system always satisfies the frequency criteria for optimal control as the polar plot for open-loop gain never crosses the unit circle, while the non-gain-insensitive system does not.

Consider a nonlinear system shown in Fig. 3-2(a) where N is of the specific type considered in Chapter II; namely, N is frequency-insensitive, single-valued, and symmetrical. The system is designed such that $n - 1$ zeros of $H_{2eq}(s)$ lie at the same places where $n - 1$ of the n open-loop poles are located. Such a system can be reduced to a simple first-order nonlinear system in series with an $(n - 1)$ st order system as shown in

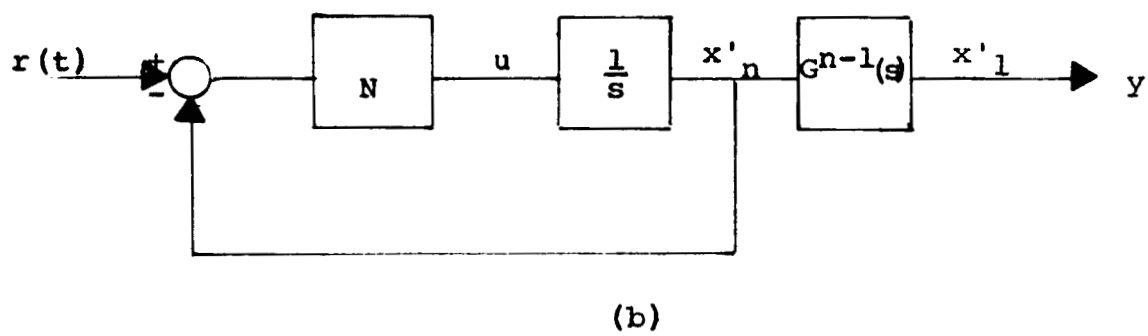
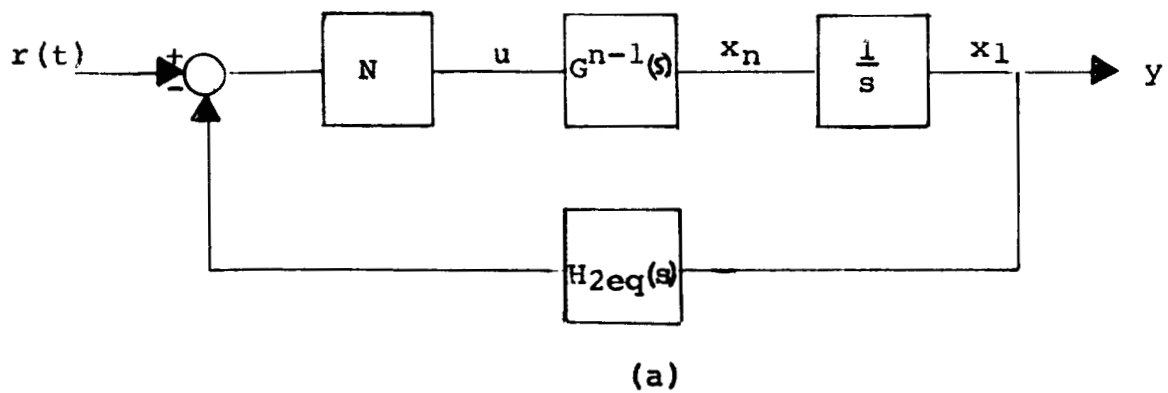


Fig. 3-2 Nonlinear Gain-Insensitive System and Modified Block Diagram

Fig. 3-2(b). It is easy to analyze such a system by graphical methods such as the like isocline method. The system designed by the non-gain-insensitive method is of n^{th} order and cannot be reduced to any such simple form and hence cannot be analyzed as easily by graphical methods.

Although the gain-insensitive method of designing a system is superior to other techniques in many respects, it is difficult to put the zeros of $H_{2eq}(s)$ exactly on top of the poles of $G(s)$. If cancellation does not take place, then the system has n poles which vary with the gain, possibly even becoming unstable if the poles are near the $j\omega$ -axis (Herring, 1967).

The results of this and the previous chapter are now used to design a system which saturates at a certain point. It was mentioned previously that all systems saturate; typical physical components having saturating characteristics are an amplifier in the forward loop and the movement of some mechanical part which is restricted to a certain range. In Chapter II it was shown that the saturating element might cause the system to oscillate if it is not located at the proper place within the loop. The locations of such elements are not controllable as they are part of the physical system.

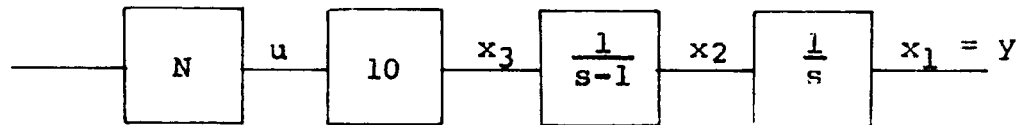
A way to prevent saturation of such an element is to control the input signal to that element; this can be done by introducing an intentional nonlinearity having a limiter type characteristic with the proper limiting values. With the introduction of such an element the system following the nonlinearity always operates in its linear region since the nonlinearity input is always restricted to the range of linear operation for the nonlinearity.

In Chapter II it was shown that if the location of the nonlinear element is at the left most end and state variable feedback is used, there are $n - 1$ zeros of $H_{2eq}(s)$ to control the plant. Thus it can be seen that if a limiter is introduced at the left end and if state variable feedback is used, then saturation in other parts of the system can be prevented and the system can be made stable for all gain, even insensitive to gain.

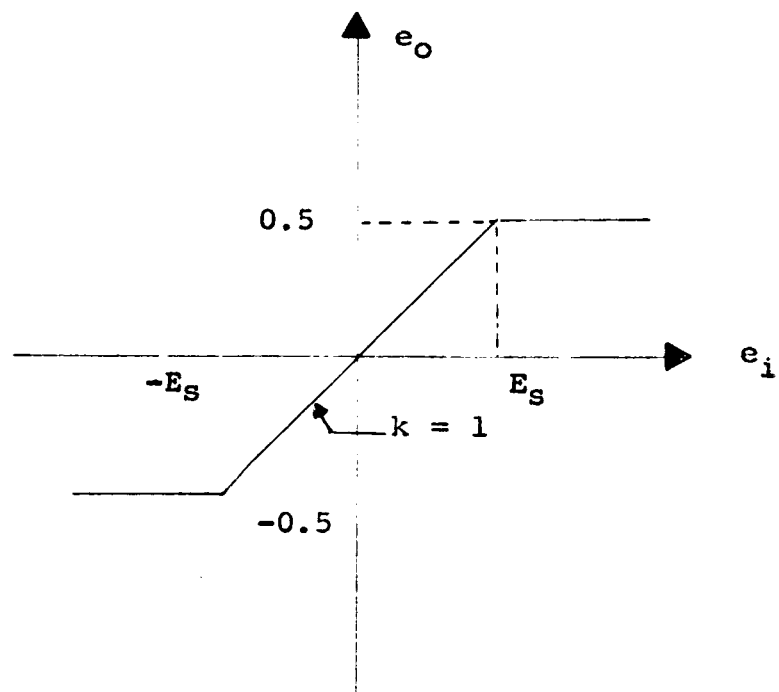
The technique is illustrated in the following example where two methods of designing the same system are presented for comparison.

Example 1

Consider the plant shown in Fig. 3-3(a) and having an intentionally introduced nonlinear element



(a)



(b)

Fig. 3-3 Plant and Characteristic of N for Example 1

of the saturation type whose characteristic is shown in Fig. 3-3(b). When operating in the linear region

$$G_p = \frac{10}{s(s - 1)}$$

and the required closed-loop transfer function is chosen to be

$$\frac{Y}{R}(s) = \frac{10}{(s + 2)(s + 5)}$$

Gain-Insensitive Design

Now feeding back x_2 to modify the plant so that $n - 1$ (1) of the open-loop poles lie at the same place as $(n - 1)$ one of the closed-loop poles, gives the modified open-loop plant, as

$$G(s) = \frac{10}{s - 1 + 10k_2'} \cdot \frac{1}{s}$$

The value of k_2' that places one of the poles of $G(s)$ at the closed-loop pole location $s = -2$ is $k_2' = .3$. Next, both x_1 and x_2 are fed back from the modified $G(s)$ to realize the desired closed-loop transfer function when operating in the linear region. By block diagram manipulation

$$\frac{Y}{R} = \frac{10}{s^2 + 2s + 10(k_2s + k_1)}$$

Equating the denominators of the required and the designed closed-loop transfer functions, k_1 and k_2 are found to be $k_1 = 1.0$ and $k_2 = 0.5$.

Non-Gain-Insensitive Design

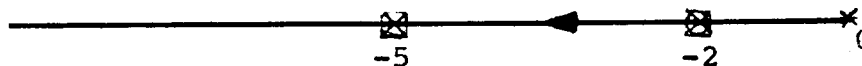
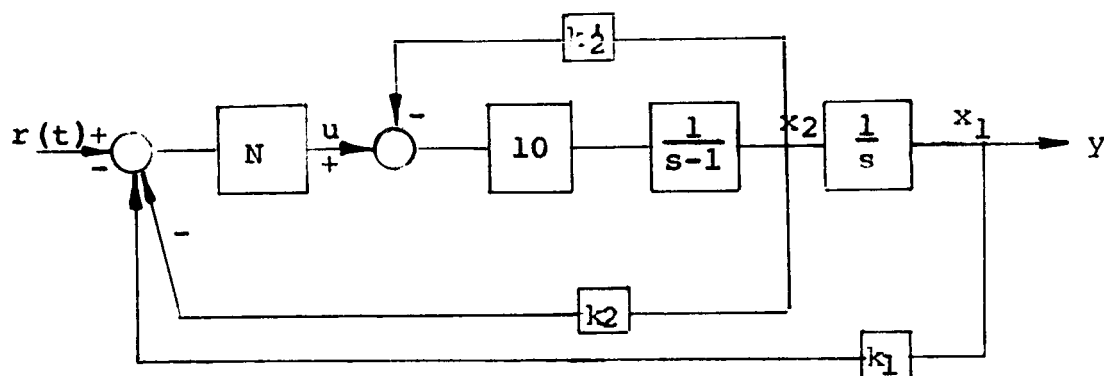
Here both x_1 and x_2 are fed back directly from $G_p(s)$. Block diagram manipulation yields

$$\frac{Y}{R} = \frac{10}{s^2 - s + 10(k_2s + k_1)}$$

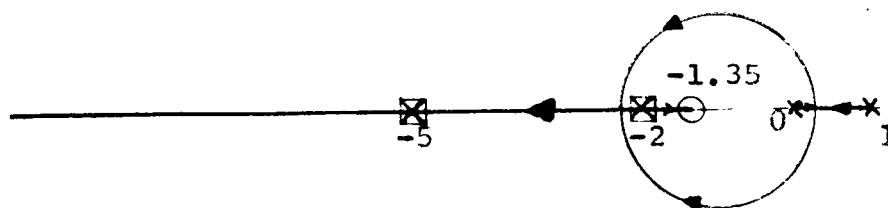
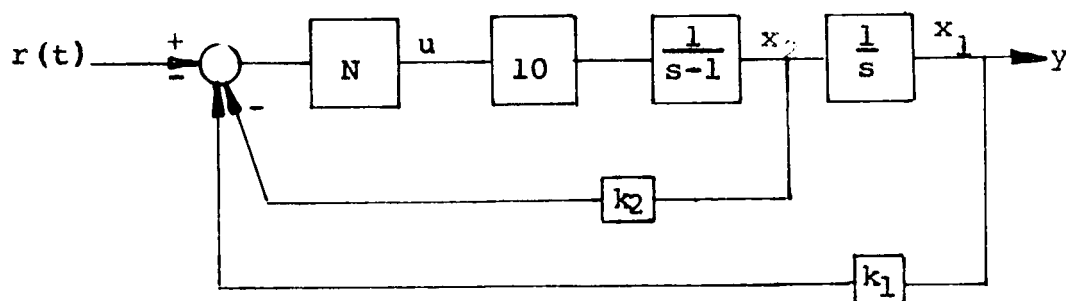
Comparing the denominator of the required and designed expression for Y/R , k_1 and k_2 are found to be $k_1 = 1.0$, $k_2 = 0.800$.

Both systems are shown with their root locus in the linear region of operation in Fig. 3-4(a) and 3-4(b). Both systems were simulated on an analog computer, and the step responses are presented in Fig. 3-5(a) and 3-5(b), respectively. It can be seen that for a step input, in the linear region of operation, both systems respond in the same way. However, when the input is increased so that the systems operate in nonlinear region of N , the non-gain-insensitive system gives an overshoot while the other system does not; in fact, the response of the gain-insensitive system does not differ very much from its response in the linear region.

The behavior of the non-gain-insensitive system in the nonlinear region can be explained as follows. Consider the characteristic of a general saturation type nonlinearity shown in Fig. 3-6. e_i is the input to the nonlinearity, e_o represents the output, and k is

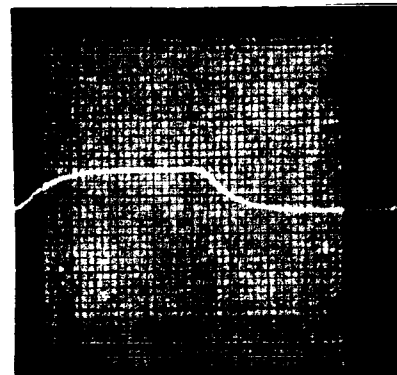
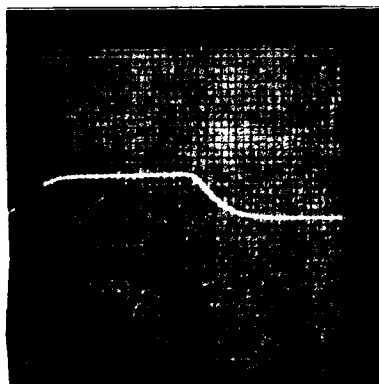


(a)

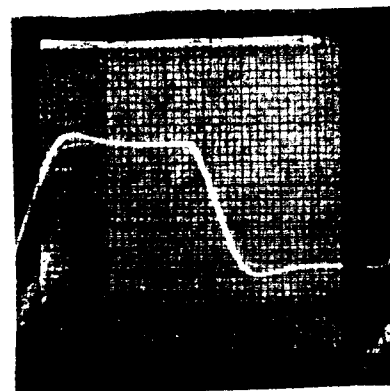
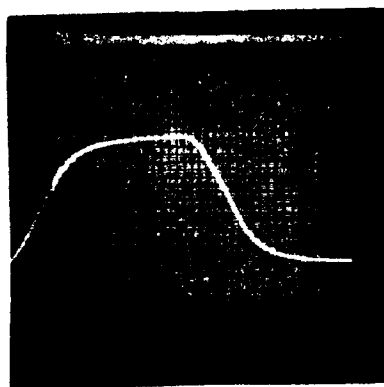


(b)

Fig. 3-4 Gain-Insensitive and Non-Gain-Insensitive Systems with Their Root Locus Sketch in the Linear Region



Operation in the Linear Region

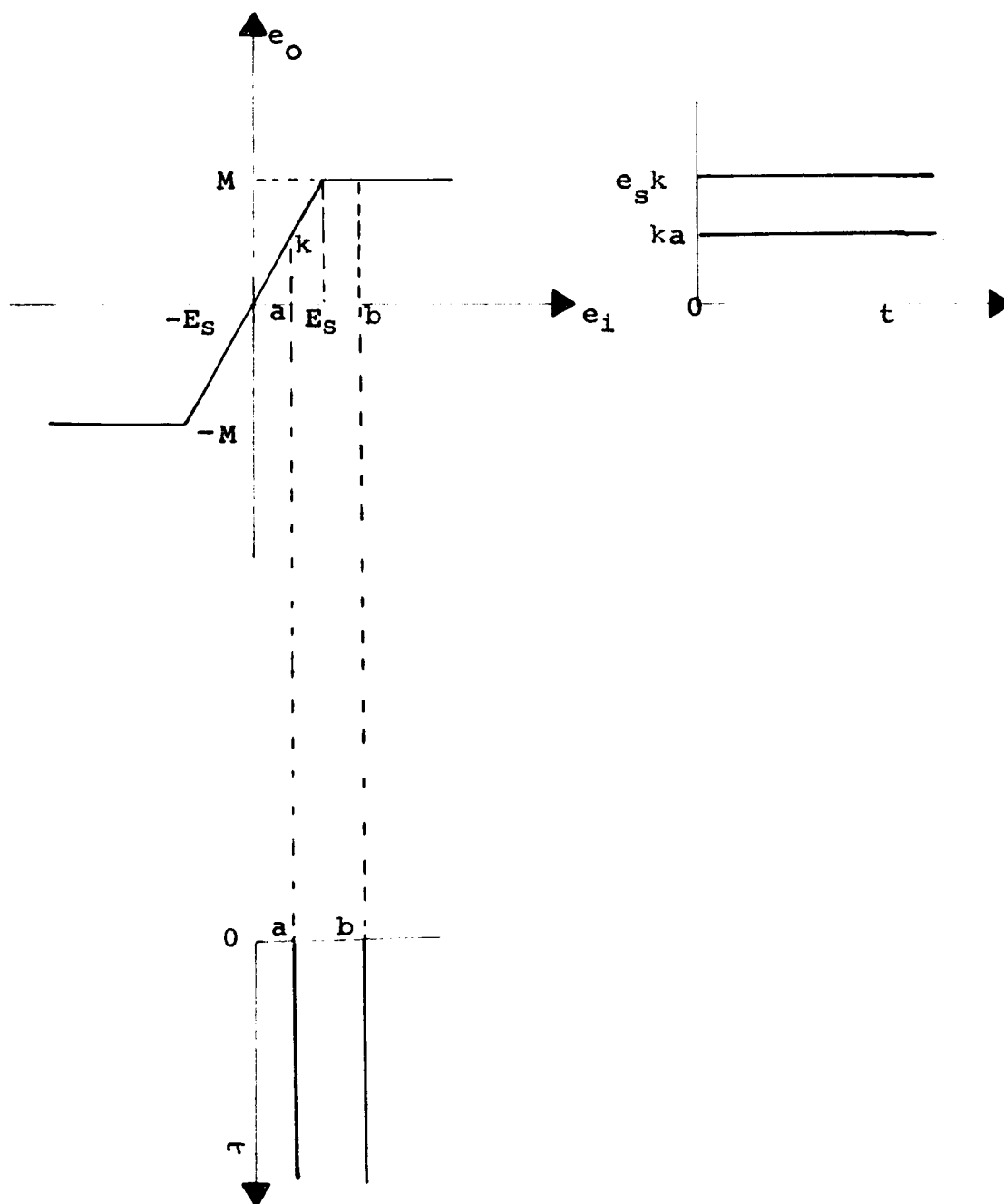


Operation in the Nonlinear Region

(a)

(b)

Fig. 3-5 Time Response for the System of Example 1



$$k' = \frac{\text{Output}}{\text{Input}}$$

Fig. 3-6 Explains the Decrease in k' When N Operates in the Nonlinear Region

the gain in the linear region of operation. When the input has a magnitude less than e_s , the output is k times the input and the equivalent gain is

$$k' = \frac{\text{output}}{\text{input}} = k$$

When $|e_i| > e_s$, the output is $\pm e_s k$ and k' becomes

$$k' = \frac{\pm e_s k}{\text{input}} < k$$

Thus it can be seen that as the input amplitude increases k_{eq}' decreases. In Example 1 when the input amplitude is increased, so that the input to N is greater than $e_s = 0.5$, k' decreases and hence the total gain in the loop decreases, causing the n poles of non-gain-insensitive system to assume a different configuration. The new closed-loop configuration can be a pair of complex conjugate poles (see root locus sketch Fig. 3-5(b)), which causes overshoot in the output of the system.

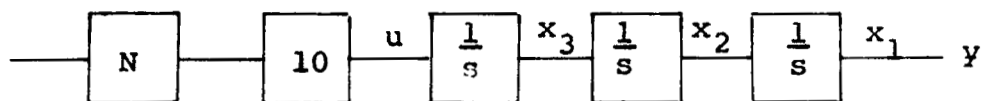
Example 2

Consider the plant shown in Fig. 3-7(a). The nonlinearity N is of the saturation type as shown in Fig. 3-7(b). In the linear region

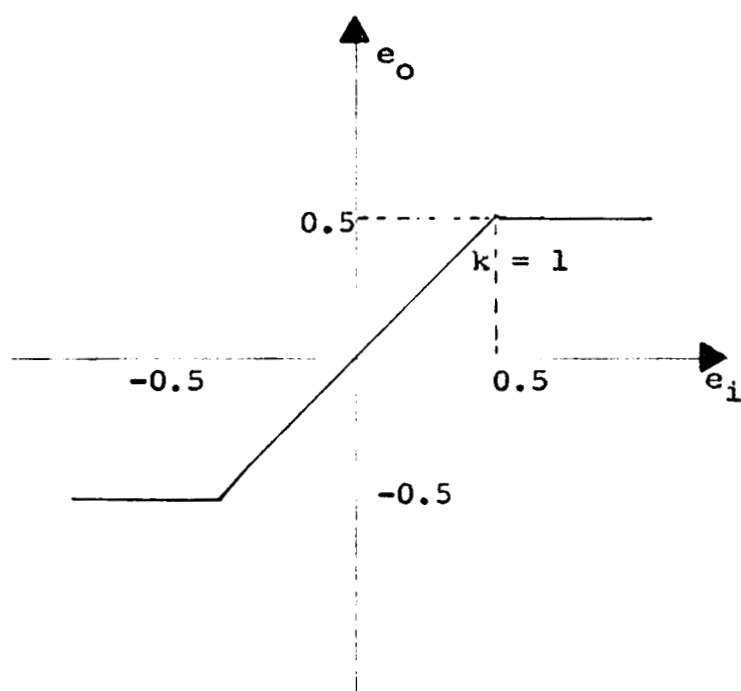
$$G_p = \frac{1}{s^3}$$

and the desired closed-loop transfer function is chosen to be

$$\frac{Y}{R}(s) = \frac{10}{(s + 10)(s^2 + s + 1)}$$



(a)



(b)

Fig. 3-7 Plant and Characteristic of N for Example 2

Two designs, gain-insensitive and non-gain-insensitive, are shown with their root locus plots for linear operation in Fig. 3-8(a) and (b), respectively. In the linear region both systems respond in the same way, but when operating in the nonlinear region, as the step-input amplitude is increased, the non-gain-insensitive system gives more and more overshoot and finally becomes unstable. This does not happen with the gain-insensitive system. The above phenomenon can again be explained by the same reasoning given in the previous example and also can be seen from the root locus diagram.

Example 3

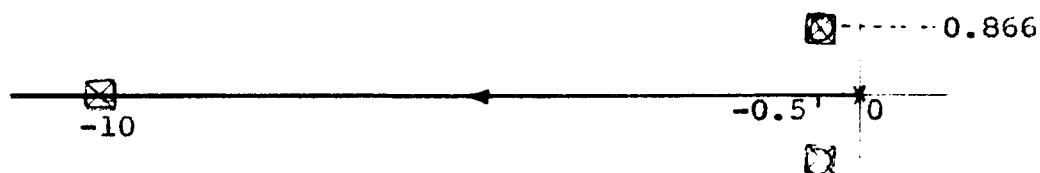
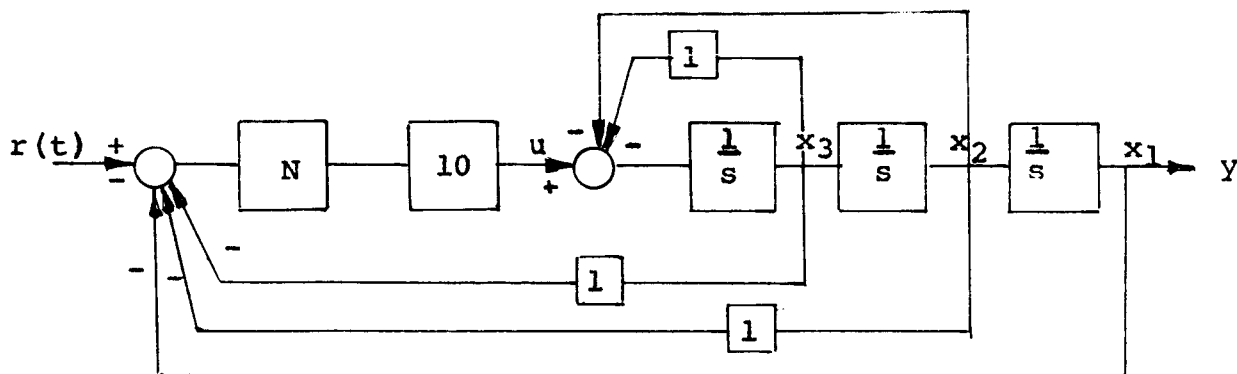
The last example has the plant shown in Fig. 3-9(a) and the nonlinearity shown in Fig. 3-9(b). In the linear region

$$G_p(s) = \frac{20}{(s^2 + 0.2s + 1)s}$$

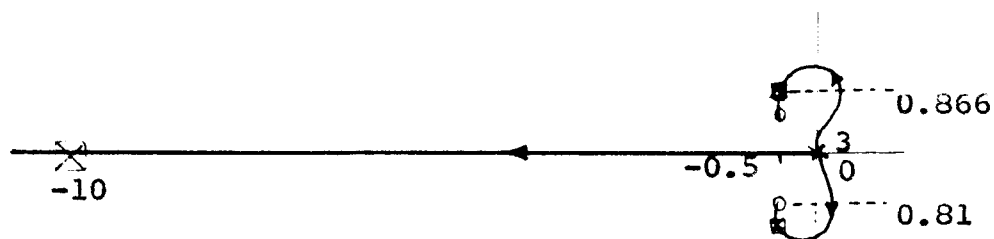
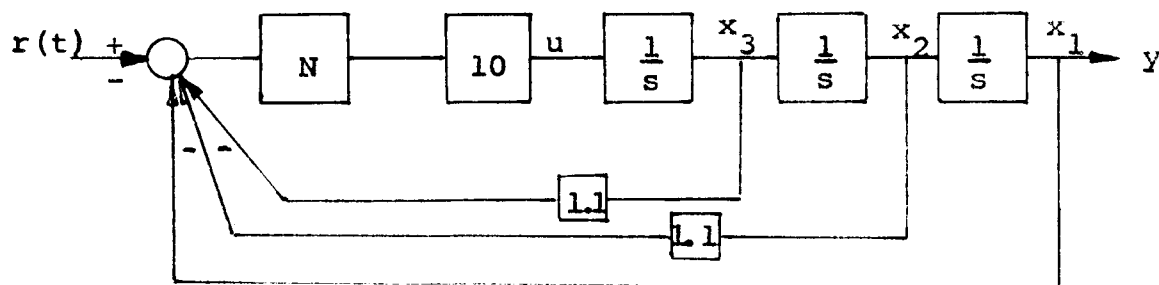
and the desired closed-loop transfer function is

$$\frac{Y}{R}(s) = \frac{20}{(s + 10)(s^2 + 0.4s + 2)}$$

Gain-insensitive and non-gain-insensitive designs are shown in Fig. 3-10(a) and 3-10(b) along with their root locus diagrams for linear operation. Both systems were simulated on the analog computer and the response to a step input is presented in Fig. 3-11(a) and 3-11(b).

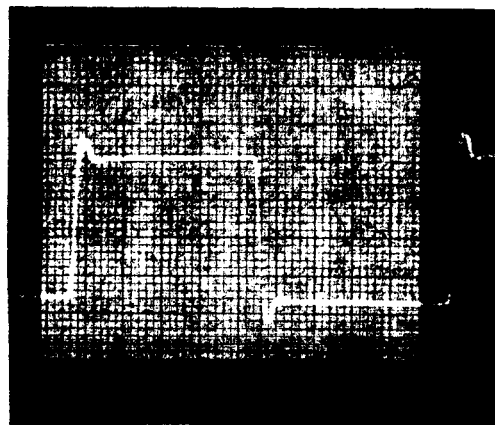
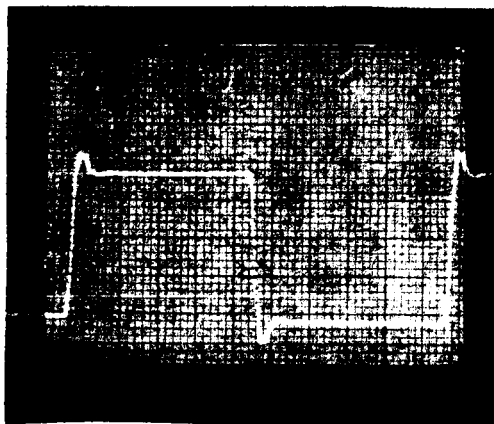


(a)

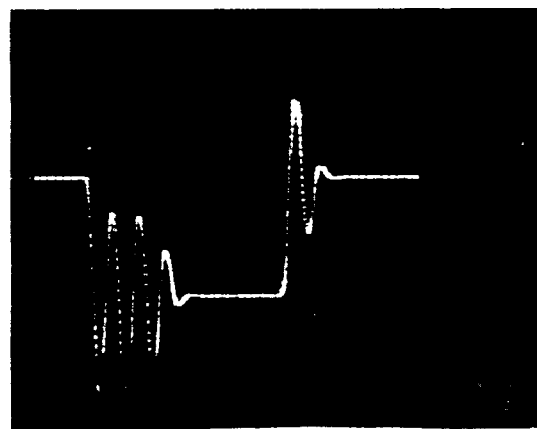
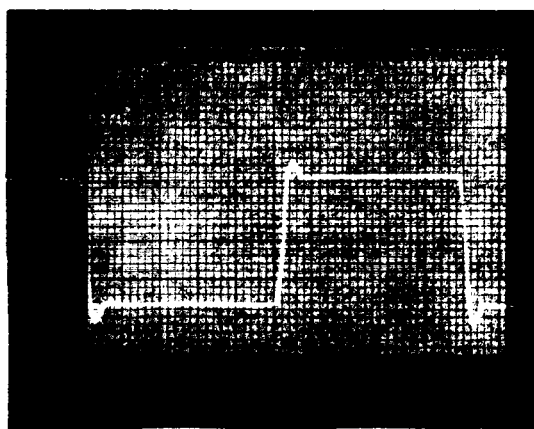


(b)

Fig. 3-8 Gain-Insensitive and Non-Gain-Insensitive Systems Along with Their Root Locus Sketch in the Linear Region



Operation in the Linear Region

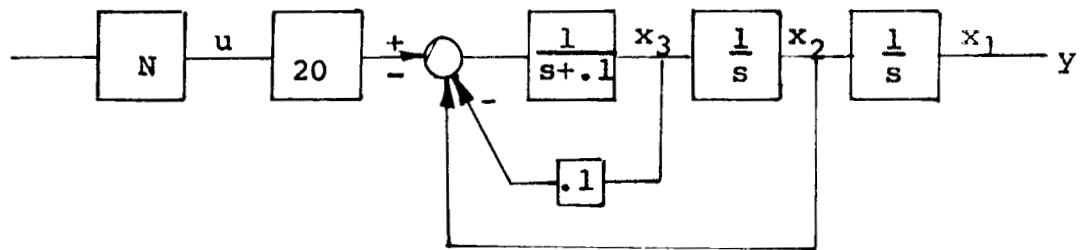


Operation in the Nonlinear Region

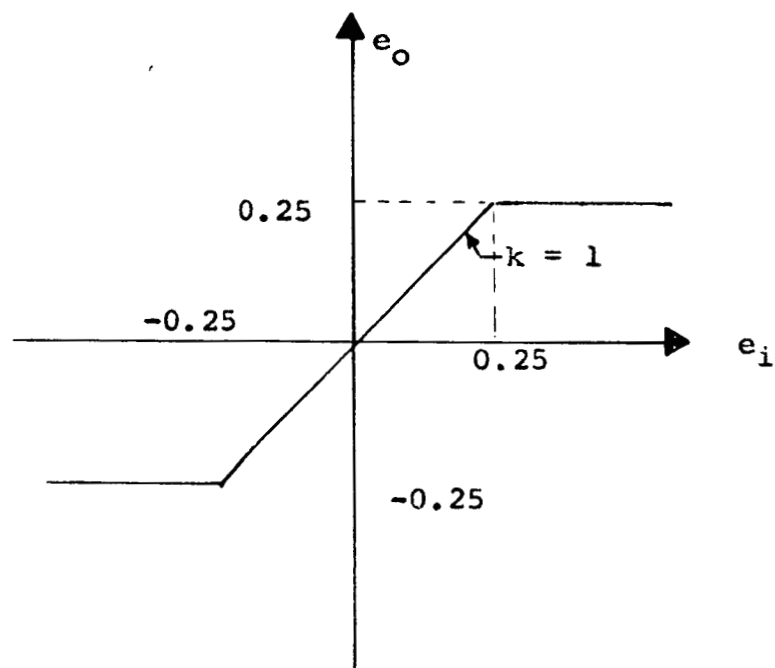
(a)

(b)

Fig. 3-8' Time Response for the System of Example 2



(a)



(b)

Fig. 3-9 Plant and Characteristic of N for Example 3

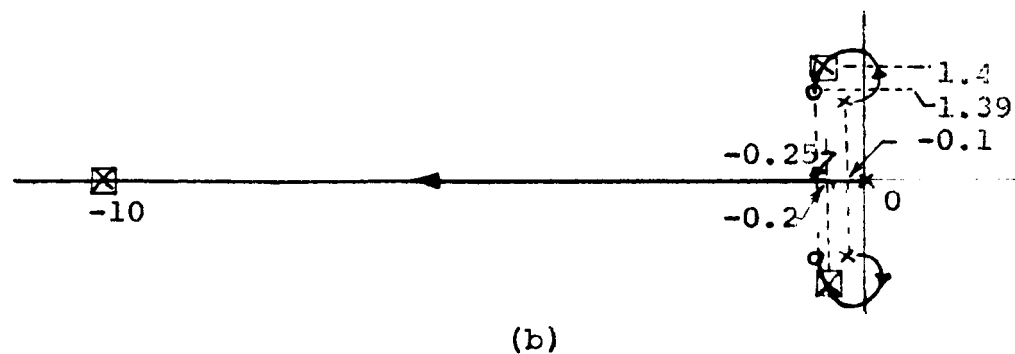
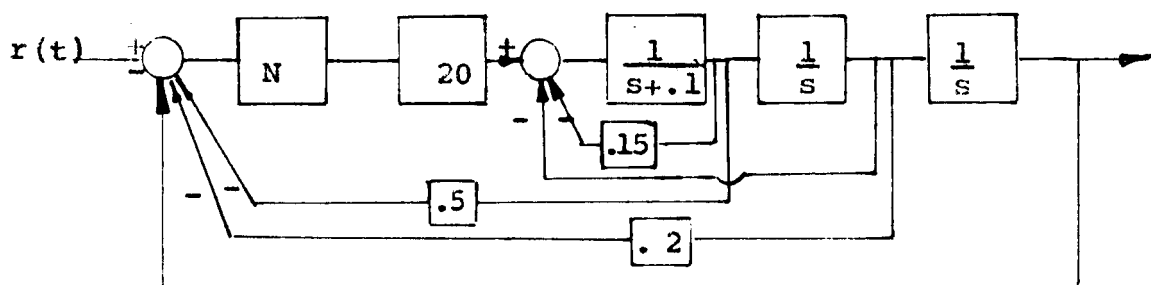
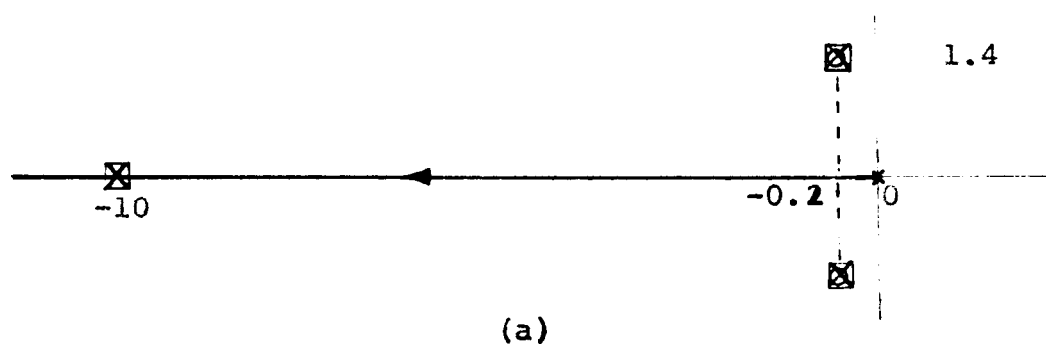
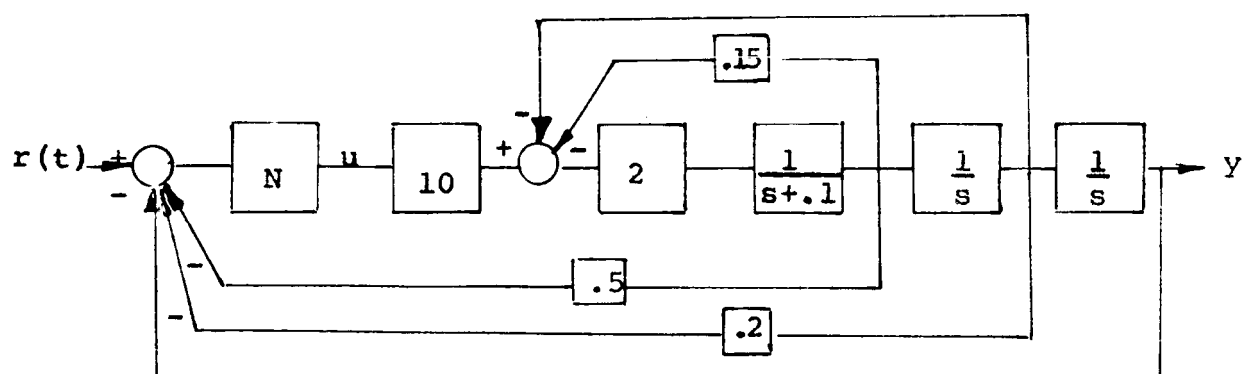
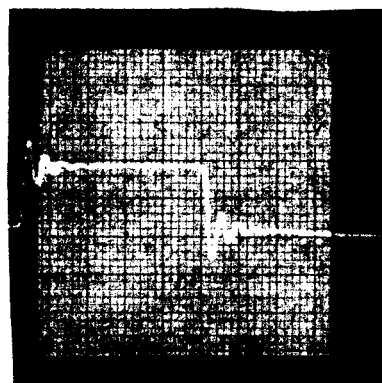
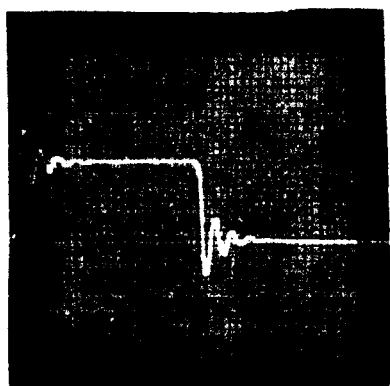
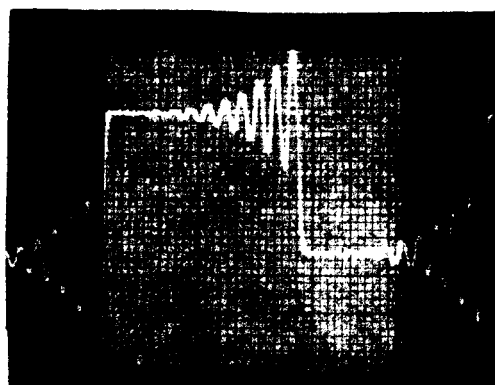
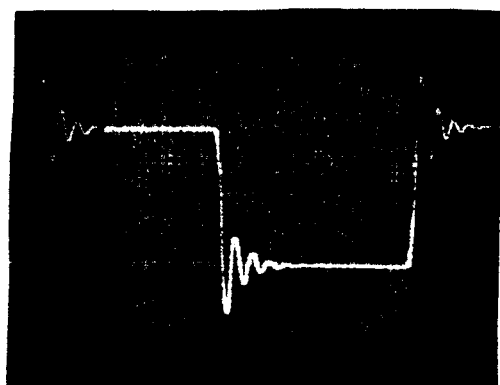


Fig. 3-10 Gain and Non-Gain-Insensitive Systems Along With Their Root Locus Sketch in the Linear Region



Operation in the Linear Region



Operation in the Nonlinear Region

(a)

(b)

Fig. 3-11 Time Response for the System of Example 3

When operating in the linear region, the response to a step input is the same for both systems. When the input is increased so that the systems operate in the region in which the nonlinearity is saturated, the results show that the non-gain-insensitive system gives more overshoot than when operating in the linear region and that the transient takes a relatively long time to die down. Also, when the magnitude of the input step to the system is increased more and more, a point is reached where there are sustained oscillations; these oscillations die down when the input magnitude is further increased. If the input amplitude is further increased, it again gives sustained oscillations as can be seen from Fig. 3-12. As in the previous examples, the response of the gain-insensitive system does not differ much from the linear response when operating in the nonlinear region.

From the above three examples, it can be seen that for the same closed-loop transfer function in the linear region, the system designed by the gain-insensitive method is absolutely stable and almost insensitive to gain; its response is good even when operating in the saturated region. For the system designed by the non-gain-insensitive method there is more overshoot and

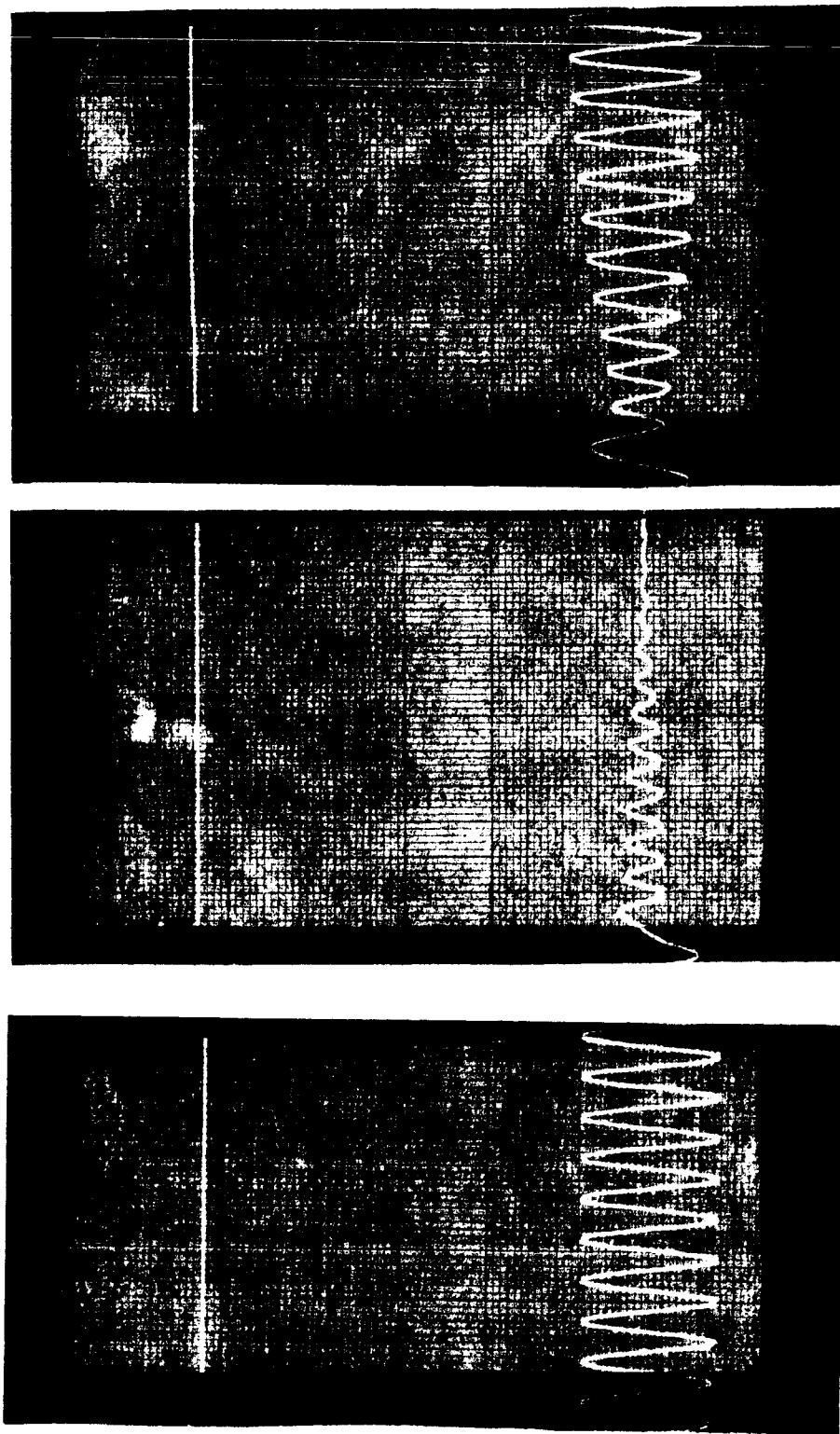


Fig. 3-12 Time Response Showing Oscillations
for Example 3

sustained oscillations if the plant is unstable or conditionally stable. Thus from the above observations it can be seen that the system stabilized by introducing an intentional nonlinearity and designed by the gain-insensitive method gives a more satisfactory performance although it increases the complexity of the system.

In the next chapter the gain-insensitive design technique is applied to a practical, high-order design problem.

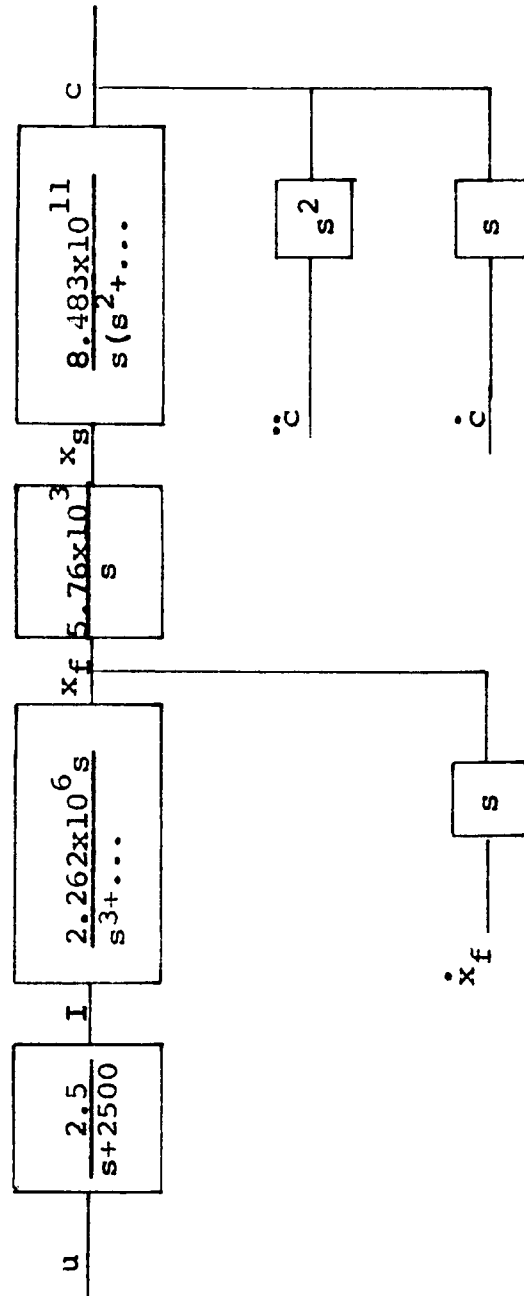
CHAPTER IV

DESIGN OF A FUEL VALVE SERVOMECHANISM

In this chapter the results of the previous two chapters (that is, an intentional nonlinearity can be introduced at the left end of the plant to prevent saturation of signals further down in the system, and using state variable feedback a system can be made absolutely stable and insensitive to gain) are applied to improve the performance of a fuel valve servomechanism for a General Electric J-85 jet engine. The engine is being used at Lewis Research Center, a NASA facility, for studying engine and inlet controls for the supersonic transport.

In order to apply the design technique it is necessary to start with a linear model of the physical system. Fig. 4-1 shows the block diagram of the 7th order linearized plant where the state variables are

- c Actuator position
- \dot{c} Actuator velocity
- \ddot{c} Actuator acceleration
- x_s Spool valve displacement
- x_f Flapper valve displacement



Third-Order Polynomial: $s^3 + 3.66937 \times 10^3 s^2 + 2.1038 \times 10^7 s + 1.7617 \times 10^{10}$

Second-Order Polynomial: $s^2 + 6.6831 \times 10^3 s + 3.28 \times 10^8$

Fig. 4-1 Linearized Plant of Physical System

\dot{x}_f Flapper valve velocity

I Torque motor current

Let $c = x_1$, $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $x_s = x_4$, $x_f = x_5$, $\dot{x}_5 = x_6$, and $I = x_7$. Then the plant can be described by 7 first-order differential equations as shown in the Appendix and can be represented by equations (Ab) and (c)

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{b}u \quad (\text{Ab})$$

$$y = \underline{c}^T \underline{x} \quad (\text{c})$$

where

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3.28 \times 10^8 & -6.68 \times 10^3 & 8.48 \times 10^{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.76 \times 10^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3.05 \times 10^6 & -2.10 \times 10^7 & -3.66 \times 10^3 & 2.26 \times 10^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2.5 \times 10^3 \end{bmatrix}$$

$$\underline{b}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2.5 \times 10^0 \end{bmatrix}$$

$$\underline{c}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the actual physical system the signals x_f , x_s , and c are limited to magnitudes less than 0.0012 inches, 0.015 inches, and 0.125 inches, respectively.

There have been at least two previous compensation schemes to improve the performance of this control system,

both of which utilized the above linear model. One scheme was to use conventional lead-lag compensation; the resulting system had a bandwidth of 220 hertz and a step response with an overshoot of 10% for small size step inputs. For input amplitudes of over 10% full scale the effects of the saturation limits caused an unsatisfactory deterioration of the response.

The second scheme utilized state variable feedback and sought to achieve a much faster response than that resulting from the lead-lag compensation. The resulting design required feedback from 5 of the 7 state variables and had a bandwidth of 700 hertz and an overshoot of less than 10% in the step response. Unfortunately, when the saturation limits on the system variables were introduced, for disturbances of any reasonable magnitude the system per cent overshoot in the transient response was excessive; and the system bandwidth decreased to approximately 100 hertz (Slivinsky, Dellner, Aparasi, 1967).

In this chapter the linearized system is first designed by the gain-insensitive method for a bandwidth of about 350 hertz and an overshoot less than 10%. Then an intentional nonlinearity of the saturation type is introduced whose saturating limits are found experimentally

on an analog computer so that the signals at x_f , x_s , and the output do not saturate when the full-scale input is applied.

Design of Gain-Insensitive System

The gain-insensitive design is carried out in three steps: selecting the desired closed-loop transfer function, modifying the plant so that 6 of the 7 closed-loop poles are achieved, and finding the feedback coefficients so that the closed-loop transfer function is realized.

As an aid in carrying out the first step one can refer to the pole-zero configuration for the original plant as shown in Fig. 4-2. Studying this plot and the normalized step-and frequency-response curves satisfying the ITAE performance index (integral of time multiplied absolute error, Graham and Lathrop, 1955) a second-order model is chosen with $\omega_n = 2250$ radians/second and $\zeta = 0.7$ to realize a bandwidth of about 350 hertz and an overshoot of less than 10%. Thus the second-order model has the transfer function

$$\left(\frac{Y}{R}\right)_{\text{model}} = \frac{5.0625 \times 10^6}{s^2 + 3.15 \times 10^3 s + 5.0625 \times 10^6} \quad (4-1)$$

The model is extended to the seventh order by choosing 2 of the seven poles to be located as in

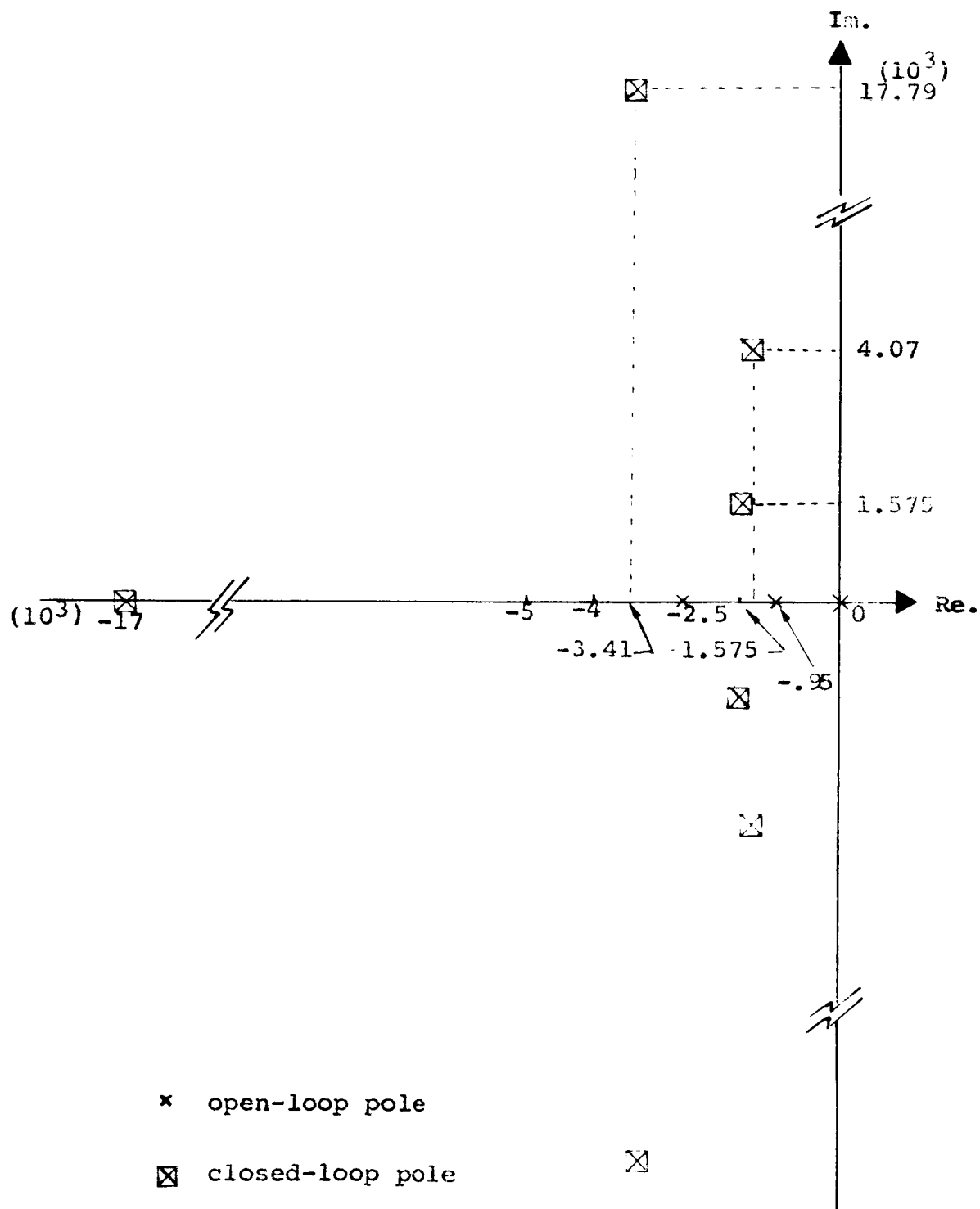


Fig. 4-2 Open-Loop and Closed-Loop Pole Location for the Linear System

Equation 4-1, 1 at the location $s = -1.7 \times 10^4$, and the remaining 4 at the same positions as the complex conjugate poles of the fixed plant. The resulting configuration is shown in Fig. 4-2, and the closed-loop transfer function is given by

$$\left(\frac{Y}{R}\right)_{\text{extended}} = \frac{5.0995 \times 10^{26}}{(s+3.34149 \times 10^3 + j1.77998 \times 10^4)} \frac{(s+1.3556 \times 10^3 + j4.07384 \times 10^3)}{(s+1.575 \times 10^3 + j1.575 \times 10^3)(s+1.7 \times 10^4)} \quad (4-2)$$

Note that Y/R approaches 1 as s approaches 0 so that that system has 0 steady-state error for step inputs.

The extended model was checked for time response and frequency response, and it was found that the results were almost the same as for the simple second-order system; i.e., the bandwidth was 350 hertz, and the overshoot was 8.4% with a rise time of about .0011 seconds.

To carry out the second step it is necessary to put $n - 1$ (6) of the open-loop poles where 6 of the closed-loop poles are located. The plant is modified such that the new open-loop transfer function is

$$G(s) = \frac{2.997 \times 10^{24}}{s(s+3.34149 \times 10^3 + j1.77998 \times 10^4)} \frac{(s+1.3556 \times 10^3 + j4.07384 \times 10^3)}{(s+1.575 \times 10^3 + j1.575 \times 10^3)} \quad (4-3)$$

This is done by feeding back the state variables x_2 through x_7 as shown in Fig. 4-3. With the help of the IBM 7072 digital computer, using the program of Melsa (1967) and the \underline{A} , \underline{b} , and \underline{c} matrices given above with the slight modification given in the Appendix, the coefficients were found to be

$$k_2' = -4.6774 \times 10^{-5}$$

$$k_3' = -1.7077 \times 10^{-9}$$

$$k_4' = 1.57057 \times 10^1$$

$$k_5' = 1.272 \times 10^1$$

$$k_6' = 4.6671 \times 10^{-3}$$

$$k_7' = -1.13557$$

and the gain k is 1.0839.

Now the modified plant is used in feeding back the variables x_1 through x_7 to realize the closed-loop transfer function given in Equation 4-2. The system is as shown in Fig. 4-4. Again, Melsa's program was used to perform the calculations, this time with the \underline{A} , \underline{b} , and \underline{c} matrices corresponding to the modified plant. These matrices are given below, and the details of the derivation of the differential equations can be found in the Appendix.

$$\underline{b}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2.70982 \times 10^2 \end{bmatrix}$$

$$\underline{c}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

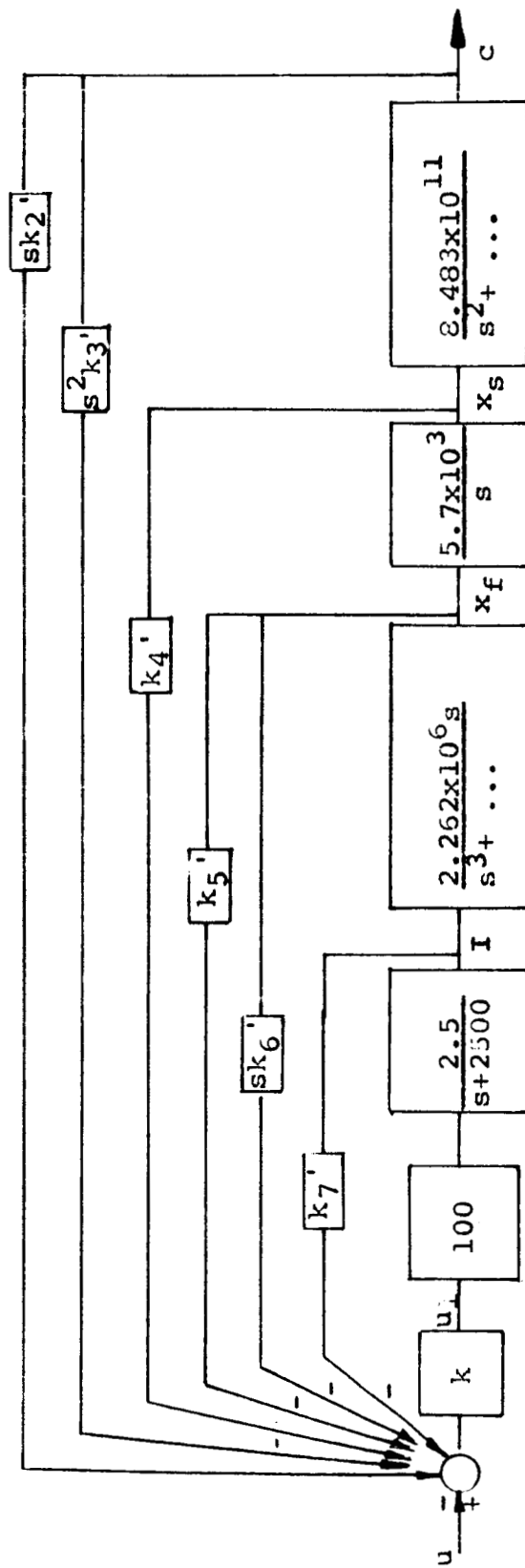


Fig. 4-3 Modification of Plant by Feeding Back Variables x_2 Through x_7

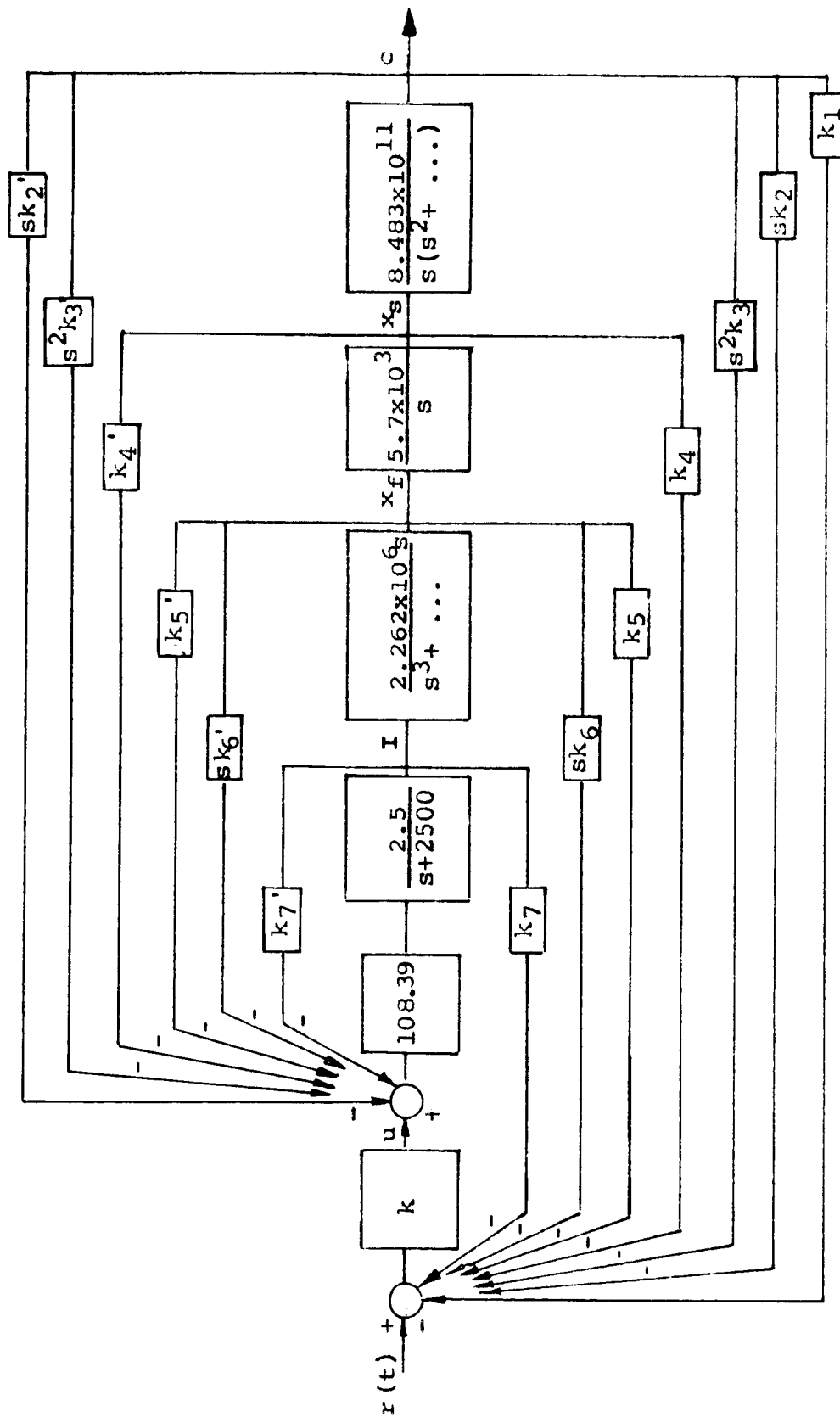


Fig. 4-4 Realization of Closed-Loop Transfer Function by Feeding Back Variables x_1 Through x_7

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.28 \times 10^8 & -6.68 \times 10^3 & -8.48 \times 10^{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.76 \times 10^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3.05 \times 10^6 & -2.10 \times 10^7 & -3.66 \times 10^3 & 2.26 \times 10^6 \\ 0 & 1.36 \times 10^{-2} & 4.62 \times 10^{-7} & -4.08 \times 10^3 & -3.44 \times 10^3 & -1.26 & 12.19 \times 10^3 \end{bmatrix}$$

The feedback coefficients for this second application of state variable feedback are given by

$$k_1 = 1.000$$

$$k_2 = 2.03317 \times 10^{-5}$$

$$k_3 = 3.06212 \times 10^{-9}$$

$$k_4 = 1.5242$$

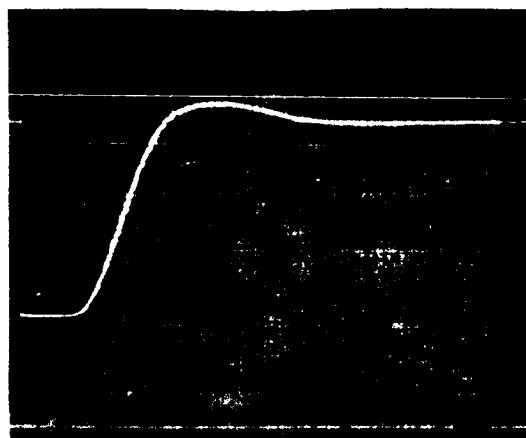
$$k_5 = 1.77896$$

$$k_6 = 3.57654 \times 10^{-4}$$

$$k_7 = 3.690275 \times 10^{-1}$$

and the gain is 1.7×10^2 .

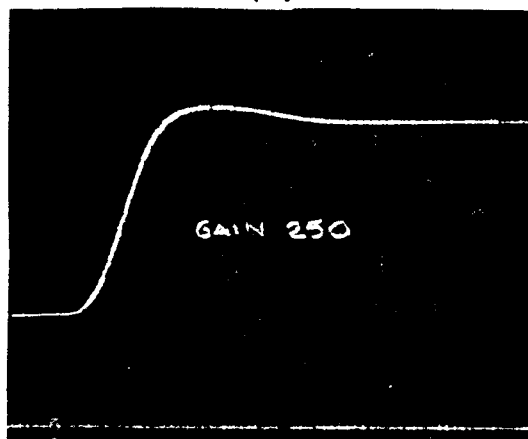
The system was simulated on an analog computer (the details of the simulation are given in the Appendix), and the time response for a step of 5 volts is given in Fig. 4-5(a) showing an overshoot of about 8.2% and a rise time of 0.00115 seconds. The feedback coefficients from different states were removed individually, and it was found that the removal of the two feedback signals from both \dot{c} and \ddot{c} does not effect



(a)



(b)



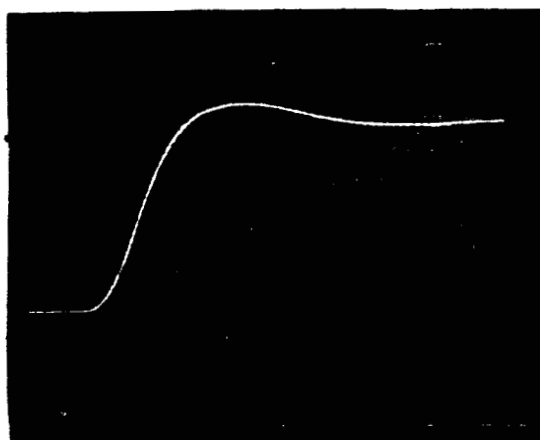
(c)

Fig. 4-5 Time Responses of the 7th Order
Linear Gain-Insensitive System

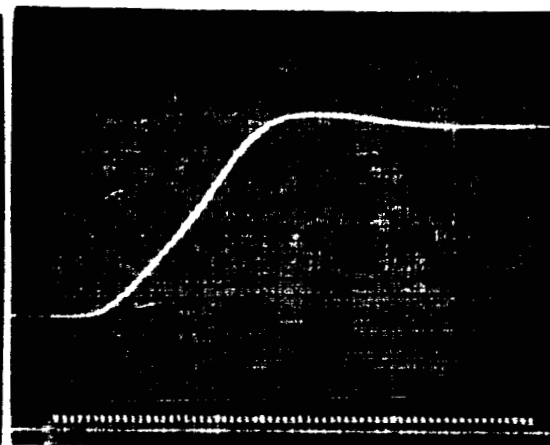
the system response very much as can be seen from Fig. 4-5(b). To check the property of gain-insensitivity the gain was varied from 100 to 250, and it was found that the effect is negligible as can be seen from Fig. 4-5(c). Thus we can conclude that the system is gain-insensitive, with a bandwidth of 350 hertz, an overshoot of 8.2% with a rise time of .00115 seconds and is unaffected by removing the feedback from \dot{c} and \ddot{c} .

In a more realistic model of the system saturation at x_s , x_f , and c must be taken into account. Here the technique of Chapters II and III is used, and an intentional nonlinearity of the saturation type is introduced, whose saturating limits were found experimentally to be ± 0.595 volts so that the signals at x_f and x_s never exceed their saturation limits.

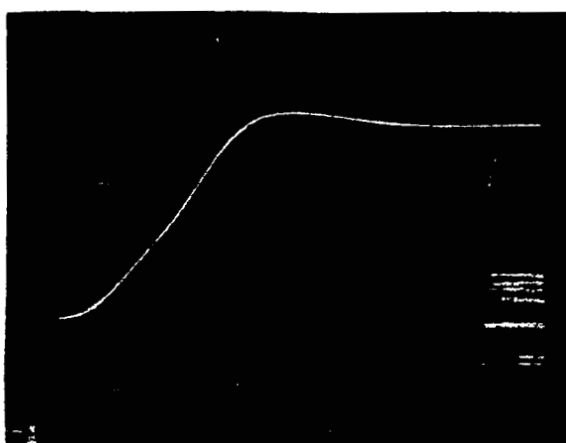
To check whether the nonlinear system is correct or not, the system response was found for the small input of 0.5 volts, and it was found to be the same as that of the linear system as shown in Fig. 4-6(a). The step response for a step size of 5 volts is shown in Fig. 4-6(b). Comparing this response with that of the linear system, one can see that the former has a large rise time because the system operates in part in the



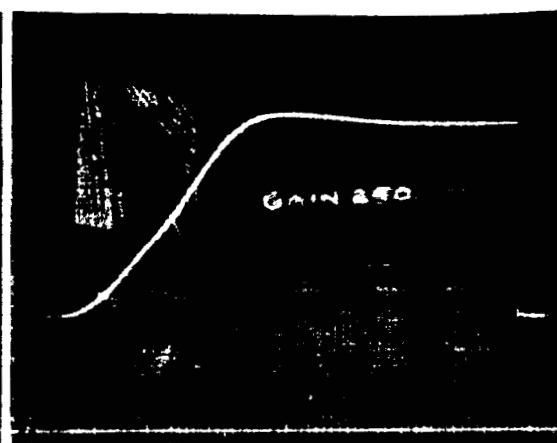
(a)



(b)



(c)



(d)

Fig. 4-6 Time Response for the Nonlinear System

saturated region. The overshoot is about the same as for the linear system.

The system response was checked with the feedback signals removed from \dot{c} and \ddot{c} , and it was found that the response is not much affected. The signals x_f and x_s do not exceed their saturating limits; and the per cent overshoot is in the same range as previously, as can be seen from the time response shown in Fig. 4-6(c). Also, the effects of varying the gain, which was varied from 100 to 255, were checked; and the response was found to be almost unaltered, as can be seen from Fig. 4-6(d). The system sensitivity was evaluated by varying the feedback coefficients by $\pm 25\%$, and it was found that this variation of the feedback coefficients does not cause any serious problems. Thus it can be concluded that the nonlinear system is insensitive to gain, variations in feedback coefficients, and the removal of the feedback signals from \dot{c} and \ddot{c} . When the input is such that the system operates in the nonlinear range, the step response is slower than that of the linear system but the per cent overshoot is almost the same.

CHAPTER V

SUMMARY AND CONCLUSIONS

The representations of linear and a certain class of nonlinear state variable feedback systems have been presented. The nonlinear system was assumed to have a single nonlinear element of the non-memory type which was symmetric and had its characteristic lying in the first and third quadrant. The G_{eq} and H_{eq} representation was used to show that the optimum location for the nonlinear element is at the left end, although stability depends on both location of the nonlinearity and the locations of the zeros of $H_{2eq}(s)$.

To ensure absolute stability for all gain, the gain-insensitive method of design was proposed; and a step-by-step procedure was presented. Systems designed by the gain-insensitive method are absolutely stable and insensitive to gain. In the case of nonlinear systems, an n^{th} order system can be reduced to a first-order nonlinear system in series with the $(n-1)^{st}$ -order nonlinear system which is easy to analyze. Also, even when working in the nonlinear region the response of the nonlinear system is not degraded as much

as that of the same system designed by non-gain-insensitive methods. Thus the linear and nonlinear gain-insensitive systems are better in certain respects than non-gain-insensitive systems.

The property of inherent saturation in a plant was discussed along with effects which may cause instability. Saturation in the fixed plant can be prevented by introducing an intentional, saturation type nonlinear element with the proper limits. By combining this idea with the gain-insensitive method using state variable feedback, a system not only can be made stable but also absolutely stable for all gain.

The technique was used in improving the response of a fuel valve servomechanism which saturates at three different points. The resulting system has a large bandwidth and a low overshoot in response to a step input when operating in the linear region; in the nonlinear region, the response was better than that achieved in two previous design attempts.

Although the method worked well in the design example, there are several things yet to be investigated in connection with the design of the fuel valve servomechanism. The sensitivity of the system can be investigated further, perhaps even incorporating sensitivity requirements as one of the design criteria.

Also, whether the system response can be improved by using the conventional series compensation in combination with the gain-insensitive design technique can be investigated. A systematic method is still not available for choosing the closed-loop transfer function so that the unavailable feedback coefficients can be made negligibly small. The technique of introducing an intentional nonlinearity has been discussed for a particular type of system. It still has to be determined whether the technique is applicable to systems having other types of nonlinearities, such as a relay with dead space.

APPENDIX

Here the derivations of three sets of the (Ab) and (c) system equations are presented for the fuel valve servomechanism. Also, details of the analog computer simulations are given for this same system.

The differential equations describing the fuel valve servomechanism are derived with the aid of the block diagram presented in Fig. A-1. Let $c = x_1$, $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $x_s = x_4$, $x_f = x_5$, $\dot{x}_5 = x_6$, and $I = x_7$. Assuming all initial conditions to be zero, the first two equations describing the plant are

$$\dot{x}_1 = x_2 \quad (A-1)$$

$$\dot{x}_2 = x_3 \quad (A-2)$$

From the figure the transfer function relating x_1 to x_4 can be used to find \dot{x}_3

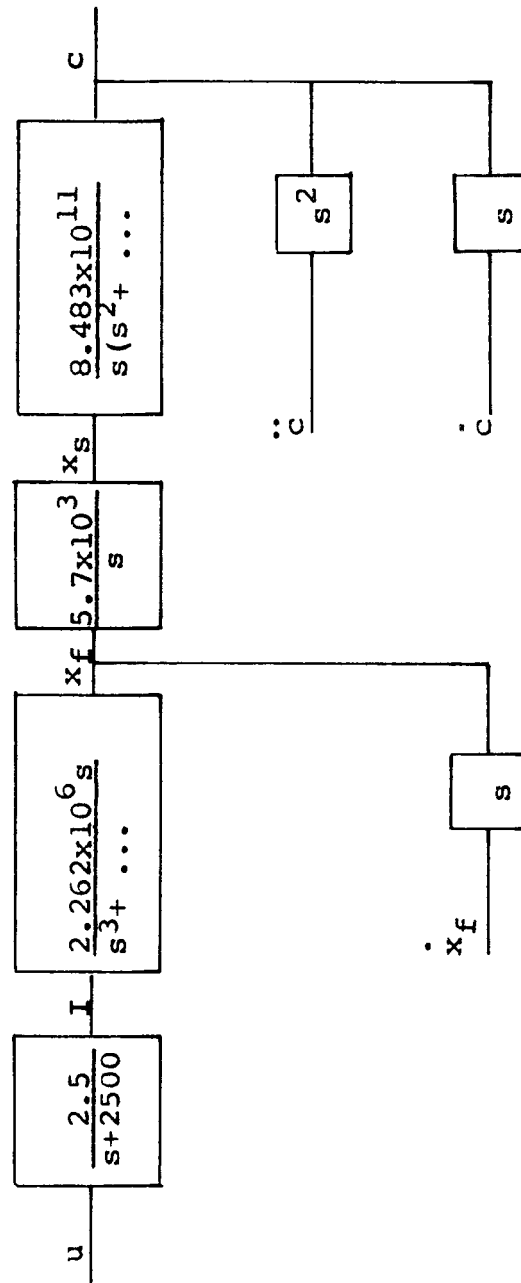
$$\frac{x_1}{x_4} = \frac{8.483 \times 10^{11}}{s^3 + 6.683 \times 10^3 s^2 + 3.28 \times 10^8 s}$$

Cross-multiplying and transferring to the time domain, one gets

$$\dot{x}_3 = -3.28 \times 10^8 x_2 - 6.683 \times 10^3 x_3 + 8.483 \times 10^{11} x_4 \quad (A-3)$$

Also from the relationship

$$\frac{x_4}{x_5} = \frac{5.769 \times 10^3}{s}$$



Third-Order Polynomial: $s^3 + 3.66937 \times 10^3 s^2 + 2.1038 \times 10^7 s + 1.7617 \times 10^{10}$

Second-Order Polynomial: $s^2 + 6.6831 \times 10^3 s + 3.28 \times 10^8$

Fig. A-1 Linearized Plant of Physical System

one gets

$$\dot{x}_4 = 5.769 \times 10^3 x_5 \quad (A-4)$$

and by definition

$$\dot{x}_5 = x_6 \quad (A-5)$$

The transfer function relating x_4 and x_7 can be used to find \dot{x}_6

$$\frac{x_4}{x_7} = \frac{2.262 \times 5.769 \times 10^9}{(s^3 + 3.669 \times 10^3 s^2 + 2.103 \times 10^7 s + 1.761 \times 10^{10})}$$

Cross-multiplying and transferring to the time domain, one gets

$$\begin{aligned} \ddot{\ddot{x}}_4 = & -3.669 \times 10^3 \ddot{\ddot{x}}_4 - 2.103 \times 10^7 \dot{\ddot{x}}_4 - 1.761 \times 10^{10} x_4 \\ & + 2.262 \times 5.769 \times 10^9 x_7 \end{aligned}$$

Substituting for \dot{x}_4 and \dot{x}_5

$$\begin{aligned} \dot{x}_6 = & -3.055 \times 10^6 x_4 - 2.103 \times 10^7 x_5 - 3.669 \times 10^3 x_6 \\ & + 2.262 \times 10^6 x_7 \end{aligned} \quad (A-6)$$

From the block diagram

$$\frac{x_7}{u}(s) = \frac{2.5}{s + 2.5 \times 10^3}$$

which gives

$$\dot{x}_7 = -2.5 \times 10^3 x_7 + 2.5u \quad (A-7)$$

Also

$$y = x_1 \quad (A-8)$$

Thus using Equations (A-1) to (A-8), the plant equations (Ab) and (c) can be written in matrix form.

For modifying the part of the plant from \dot{c} to u_1 as shown in Fig. 4-3, the required equations (Ab) and (c) can be found as follows. The equations for \dot{x}_2 , \dot{x}_3 , \dot{x}_4 , \dot{x}_5 , and \dot{x}_6 are the same as those of Equations (A-2, 3, 4, 5, and 6). From Fig. 4-3

$$\frac{x_7}{u_1} = \frac{2.5 \times 10^2}{s + 2500}$$

Cross-multiplying and transforming to the time domain, one gets

$$\dot{x}_7 = -2.5 \times 10^3 x_7 + 2.5 \times 10^2 u_1 \quad (\text{A-9})$$

and

$$y = x_2 \quad (\text{A-10})$$

Thus the modified matrices are

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -3.2 \times 10^8 & -6.5 \times 10^3 & 8.4 \times 10^{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 5.7 \times 10^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3.0 \times 10^6 & -2.1 \times 10^7 & -3.6 \times 10^3 & 2.2 \times 10^6 \\ 0 & 0 & 0 & 0 & 0 & 2.5 \end{bmatrix}$$

$$\underline{b}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2.5 \times 10^2 \end{bmatrix}$$

$$\underline{c}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To realize the closed-loop transfer function by feeding back all the variables, the differential

equations describing the modified plant are used. The differential equations for $\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5$, and \dot{x}_6 are the same as Equations (A-1, 2, 3, 4, 5, and 6), respectively. Again, from the block diagram shown in Fig. 4-3

$$\dot{x}_7 = -2.5 \times 10^3 x_7 + 2.5 \times 10^2 u_1$$

Substituting for u_1 in the above equation gives

$$\begin{aligned} \dot{x}_7 = & 1.3675 \times 10^{-2} x_2 + 4.6277 \times 10^{-7} x_3 \\ & -4.0828 \times 10^3 x_4 - 3.4700 \times 10^3 x_5 \\ & -2.1923 \times 10^3 x_7 + 2.70982 \times 10^2 u \end{aligned} \quad (\text{A-11})$$

Equations (A-1, 2, 3, 4, 5, 6, 11, and 8) are sufficient to describe the modified plant in matrix form to be used on the digital computer.

Details of the Analog Computer Simulations

To evaluate the designed gain-insensitive linear and nonlinear systems an analog computer of $\pm 100\text{v.}$ was used. The systems were simulated using the differential equation approach. The different variables were scaled using the following scale factors

$x_1 (2 \times 10^2)$	$x_2 (1)$	$x_3 (2.35 \times 10^3)$	$x_5 (10^4)$
$x_6 (1)$	$x_7 (2 \times 10^3)$	$u (2 \times 10^3)$	$r (4 \times 10^2)$

The limits of the saturating states x_1, x_4 , and x_5 have the magnitudes 5 volts, 35 volts, and 12 volts, respectively. The scaled differential equations are as follows:

$$\dot{x}_1 = 2 \times 10^2 x_2$$

$$\dot{x}_2 = 10^4 x_3$$

$$\dot{x}_3 = -3.28 \times 10^4 x_2 - 6.683 \times 10^3 x_3 + 3.60978 \times 10^4 x_4$$

$$\dot{x}_4 = 1.3557 \times 10^3 x_5$$

$$\dot{x}_5 = 10^4 x_6$$

$$\dot{x}_6 = -1.3 \times 10^3 x_4 - 2.1030 \times 10^3 x_5 - 3.669 \times 10^3 x_6 + 1.131 \times 10^3 x_7$$

$$\dot{x}_7 = 2.7350 \times 10^1 x_2 + 9.2554 x_3 - 3.4747 \times 10^3 x_4 - 6.894 \times 10^2 x_5 \\ - 2.5294 \times 10^3 x_6 - 2.1923 \times 10^3 x_7 + 4.60669 \times 10^4 u$$

$$u = (5r(t) - 2 \times 10^{-3} \underline{k}^T \underline{x})$$

The feedback coefficients are $k_1 = 10$, $k_2 = 4.066 \times 10^{-2}$, $k_3 = 6.124 \times 10^{-2}$, $k_4 = 1.297188$, $k_5 = 3.55792 \times 10^{-1}$, $k_6 = 7.15308 \times 10^{-1}$, and $k_7 = 3.690275 \times 10^{-1}$.

In order to facilitate the recording of step responses, the system was time-scaled by the factor 10^4 which gives the new differential equations

$$\dot{x}_1 = 0.02 x_1$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -3.28 x_2 - 0.6683 x_3 + 3.60978 x_4$$

$$\dot{x}_4 = 0.13557 x_5$$

$$\dot{x}_5 = x_6$$

$$\dot{x}_6 = -0.13 x_4 - 0.2103 x_5 - 0.3669 x_6 + 0.1131 x_7$$

$$\dot{x}_7 = 0.002735 x_2 + 0.00093 x_3 - 0.34747 x_4 - 0.06894 x_5 \\ - 0.25294 x_6 - 0.21923 x_7 + 4.60669 u$$

Using the above equations and the feedback coefficients the system circuit diagram is formed as

shown in the Fig. A-2 for the linear system. For the nonlinear system an intentional nonlinearity is introduced, whose characteristic is shown in Fig. A-3 along with the diode bridge to realize it.

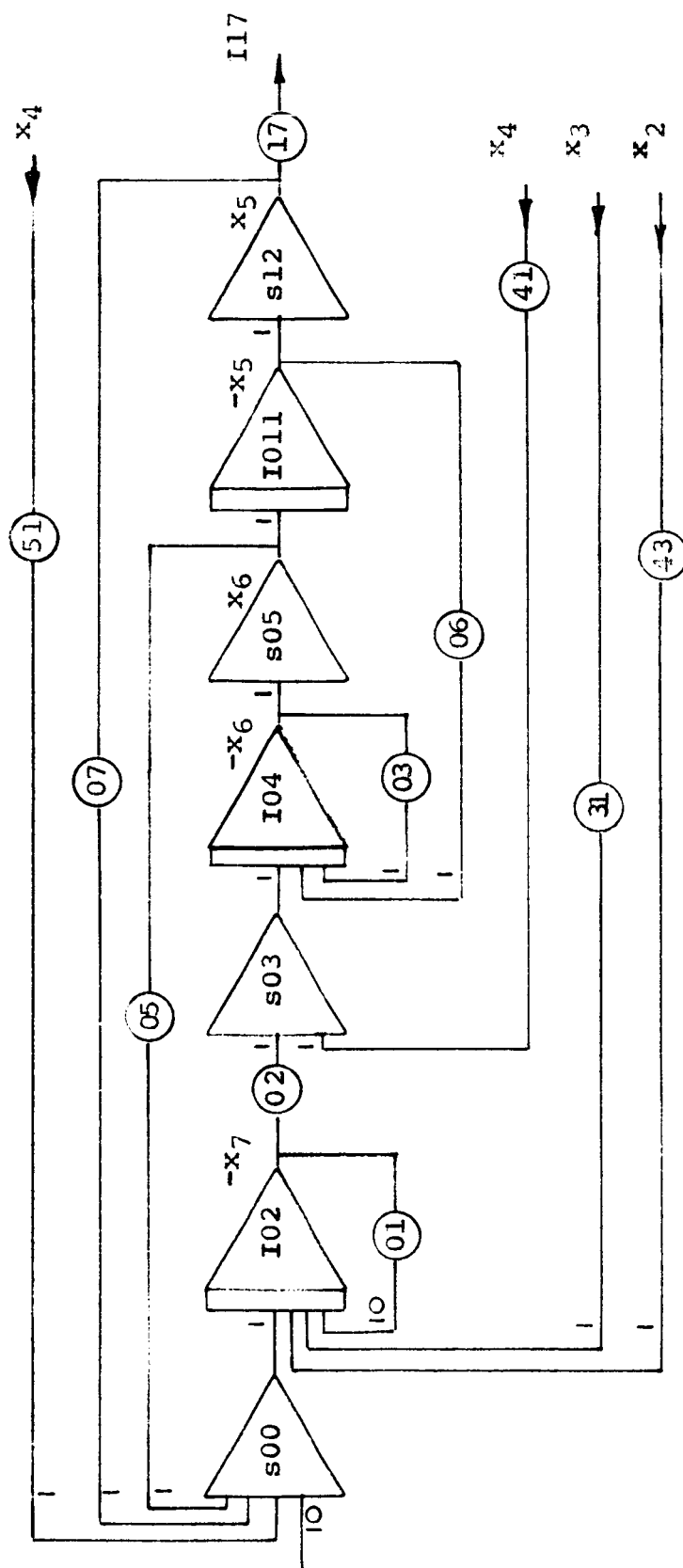


Fig. A-2 Analog Computer Wiring Diagram
for Simulation (Continued on Next Page)

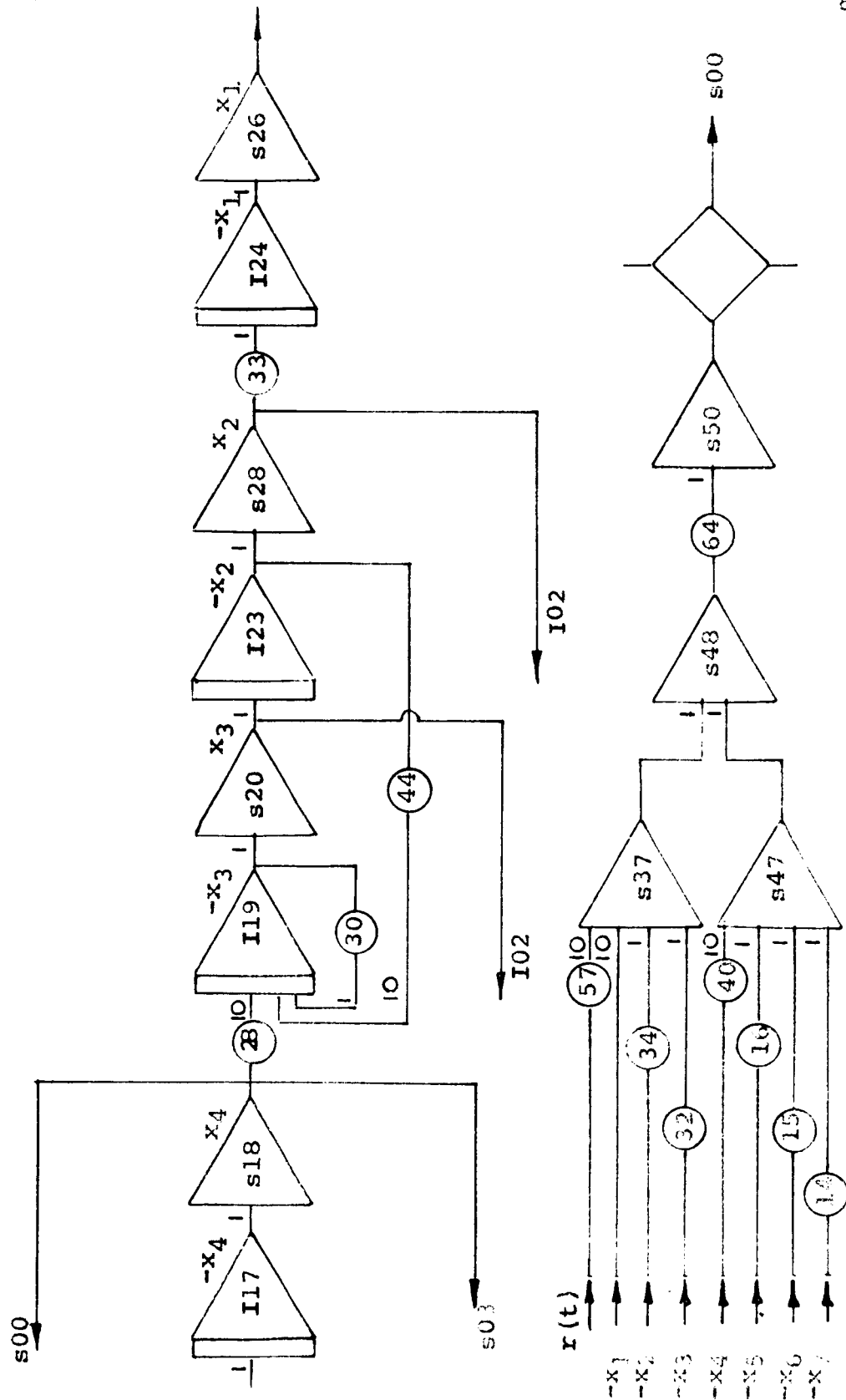


Fig. A-2 (Continued)

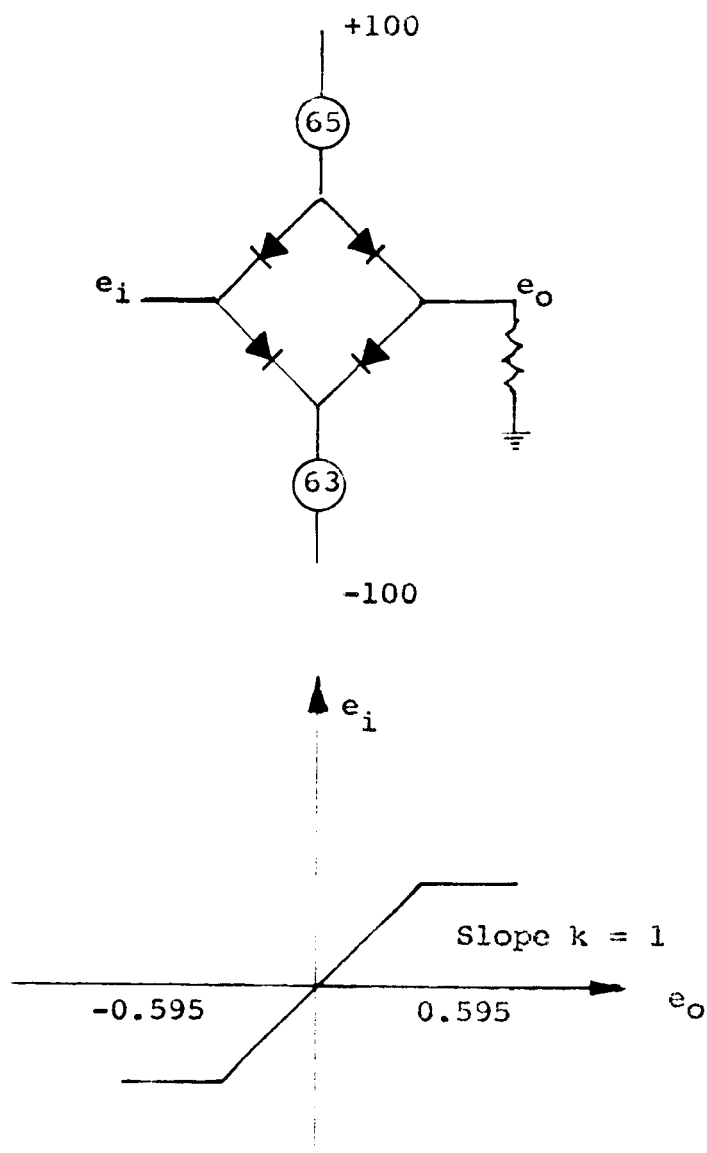


Fig. A-3 Details of the Bridge Circuit Realizing the Limiter and its Input-Output Characteristic

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